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On the Vibrations in the Field Round a Theoretical Hertzian Oscillator

Karl Pearson and Alice Lee

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V. *On the Vibrations in the Field round a Theoretical Hertzian Oscillator.*

By Karl Pearson, *F.R.S., and* Alice Lee, *B.A., B.Sc., University College, London.*

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[Plates 1–7.]

Contents.

(1.) Although Hertz realised very fully* that his oscillator did not give "perfectly regular and long continued sine-oscillations," and although Bjerknes† determined so long ago as 1891 the general form of the damping, it does not appear that Hertz's original investigation of the nature of the vibrations in the field round one of his oscillators has hitherto been modified. Indeed, his diagrams of the wave motion have been copied into more than one text-book,‡ and have usually been taken to represent what actually goes on in the surrounding field. Actually not only the diagrams, but

* "On very rapid Electric Oscillations," 'Wied. Annal.,' vol. 31, p. 421, 'Electric Waves,' p. 49.

† 'Wied. Annal.,' vol. 44, pp. 74, 513–526. The damping of the oscillation in four or five periods is very marked.

‡ For example, Andrew Gray, 'Absolute Measurements in Electricity and Magnetism,' vol. 2, p. 734.

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a good deal of HERTZ's original theory of interference requires modification, if we are to obtain quantitative accordance between theory and experiment. The object of the present paper is to give a fuller theory of the nature of the vibrations in the field round a typical Hertzian oscillator.

(2.) Assuming that BJERKNES' experiments have shown that the Hertzian oscillator vibrates very nearly according to the type:

$$C^{-p_1 t} \sin (p_2 t + \alpha),$$

we have to find a solution for the equations for the disturbance in the surrounding field on this basis.

Using HERTZ's notation* for the case of symmetry about the axis of z, then if Z and R be the components of electric force parallel and perpendicular to this axis at a point whose coordinates are z and $\rho = \sqrt{x^2 + y^2}$, and P the component of magnetic force perpendicular to the meridian plane through the same point, we have to find a solution of

$$a^2 \frac{d^2\psi}{dt^2} = \nabla^2 \psi \quad \text{. . . . .} \quad \text{(i)},$$

suitable to the initial and boundary conditions. If this be done,

$$Z = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\psi}{d\rho}\right), \qquad R = -\frac{1}{\rho}\frac{d}{dz}\left(\rho \frac{d\psi}{d\rho}\right), \quad \text{. . . . . .} \quad \text{(ii)},$$

$$P = \frac{a}{\rho}\frac{d}{dt}\left(\rho \frac{d\psi}{d\rho}\right) \quad \text{. . . . .} \quad \text{(iii)}.$$

The component of magnetic force parallel to z is zero.

Now if we suppose that at some distance from its centre the oscillator may be looked upon as a "double point" or as producing an oscillation, which has a very small range l, and with poles having $\pm$ E electricity at the maximum, then we may write for ∇^2,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right).$$

Assume $\psi = f_1(r) e^{pt}$, and we find

$$a^2 p^2 f_1 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df_1}{dr}\right).$$

Writing $r' = apr$, and $f_2 = r' f_1$, we have

$$f_2 = \frac{d^2 f_2}{dr'^2}.$$

* See 'Electric Waves,' English edition, p. 140.

Hence,

$$f_2 = A_1 e^{r'} + A_2 e^{-r'},$$

where A_1 and A_2 are constants and

$$\psi = \frac{A_1 e^{p(t+ar)} + A_2 e^{-p(t-ar)}}{apr}.$$

Take only an *outgoing* wave and write $p = -p_1 + p_2\sqrt{-1}$, hence

$$\psi = \frac{B}{r} e^{-p_1(t-ar)} \sin p_2 (t - ar) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iv})$$

where t must be $> ar$, or $\psi = 0$.

$1/a$ is clearly the wave velocity v. Take $p_2 = \frac{2\pi}{\lambda} v$, then

$$\psi = \frac{B}{r} e^{-p_1(t-ar)} \sin \frac{2\pi}{\lambda}(vt - r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v}).$$

For typical oscillators $2\pi p_1/p_2$ seems to vary from ·3 to ·5, hence $p_1 =$ ·3 to ·5 $\times v/\lambda$, or we see that if r be small as compared with λ, then

$$\psi = \frac{B}{r} e^{-p_1 t} \sin p_2 t.$$

Now, if X, Y, Z be the three components of electric force, we easily find

$$X = -\frac{d}{dx}\left(\frac{d\psi}{dz}\right), \qquad Y = -\frac{d}{dy}\left(\frac{d\psi}{dz}\right), \qquad Z = -\frac{d}{dz}\left(\frac{d\psi}{dz}\right),$$

or they are the three components of a "potential function"

$$V = Be^{-p_1 t} \sin p_2 t \frac{d}{dz}\left(\frac{1}{r}\right).$$

Take $B = -El$, and we see that V is the potential* due to a "double point" of moment oscillating between $+Ele^{-p_1 t}$ and $-Ele^{-p_1 t}$; thus the maximum charges rapidly diminish with the time. In fact, we have a system entirely analogous with that of HERTZ, except for this rapid diminution of the maximum charges with the time. It is in this running down of the maximum charges that the damping effect of the oscillator consists.

(3.) We shall now proceed to find the forces.

In the first place let us find the value of $\rho\, d\psi/d\rho$, which, following HERTZ, we will term Q. Then the components of electric and magnetic force can all be found by simple differentiation of Q, *i.e.*,

$$Z = \frac{1}{\rho}\frac{dQ}{d\rho}, \qquad R = -\frac{1}{\rho}\frac{dQ}{dz}, \quad \text{and} \quad P = \frac{a}{\rho}\frac{dQ}{dt} \quad . \quad . \quad . \quad . \quad (\text{vi}).$$

* To use HERTZ's language, see 'Electric Waves,' p. 142, and compare MAXWELL, vol. 1, § 129.

We have

$$Q = \rho \frac{d\psi}{d\rho} = \rho \frac{d\psi}{dr}\frac{dr}{d\rho} = \frac{\rho^2}{r}\frac{d\psi}{dr}$$

$$= -\frac{\rho^2 \mathrm{E}l}{r}\frac{d}{dr}\left\{\frac{e^{-p_1(t-ar)}}{r}\sin p_2(t-ar)\right\}$$

$$= -\frac{\rho^2}{r^2}\mathrm{E}le^{-p_1(t-ar)}\left\{\left(p_1 a - \frac{1}{r}\right)\sin p_2(t-ar) - ap_2\cos p_2(t-ar)\right\}$$

Now put $p_1 = \eta \sin \chi$, $p_2 = \eta \cos \chi$; then if $\rho = r\sin\theta$, we have

$$Q = \mathrm{E}lap_2 \sin^2\theta e^{-p_1(t-ar)}\left\{\frac{\cos[p_2(t-ar)+\chi]}{\cos\chi} + \frac{\sin p_2(t-ar)}{rap_2}\right\} \quad . \quad . \quad \text{(vii)}.$$

In our notation HERTZ finds*

$$Q = \mathrm{E}lap_2 \sin^2\theta \left\{\cos p_2(t-ar) + \sin\frac{p_2(t-ar)}{rap_2}\right\}. \quad . \quad . \quad . \quad \text{(viii)}.$$

It is clear that the damped wave train, such as actually occurs with the Hertzian oscillator, makes very considerable changes in the form of Q. Thus:

(a.) As we might expect the damping factor $e^{-p_1 t}$ is introduced, or rather a damping factor $e^{-p_1(t-t_r)}$ where t_r is the time at which the disturbance reaches a distance r from the centre of the oscillator. Clearly the factor $e^{p_1 ar}$ will sensibly modify the form of the lines of electric force obtained by tracing the curves

$$Q = \text{constant}.$$

(b.) The first term in the curled brackets is also sensibly modified both as to amplitude and phase.

The reader must not imagine that the difference between the formulæ (vii) and (viii) marks as soon as the vibrations are set up a great difference in the lines of electric force. All we contend for is that it marks a sensible difference in the shape after a few periods, and that this difference increases with the length of time and the distance of the part of the field considered from the oscillator. In fact loops which would remain at the end of each period finite according to formula (viii), have to shrivel up into points and disappear from the field altogether.

Suppose $2\pi p_1/p_2 = \cdot 4$, than $\tan\chi = \cdot 2/\pi$, and we find $\chi = 3° \, 38' \, 33''\cdot 3$, while $\sec\chi = 1\cdot 002{,}0244$. Hence, the amplitude of the cosine term is increased by about 2 per thousand, and the phase by 3° to 4°. It is the factor $e^{p_1 ar}$, however, which produces most sensible change. For $p_1 a = \frac{p_1 \,(\text{period})}{(\text{wave length})} = \cdot 4/\lambda$, if λ be the wave length. Now, if λ be 9·60 metres,† which was about its value for one of HERTZ's oscillators

* It seems better to write $p_2(t-ar)$ than $p_2(ar-t)$ with HERTZ, because ar must be less than t, or $Q = 0$, 'Electric Waves,' p. 142.

† HERTZ's λ is $\frac{1}{2}$ (wave-length). Considerable confusion has arisen from this in various translations of his papers. He assumes $\lambda = 4\cdot 8$ metres, $i.e.$, a wave-length $= 9\cdot 6$.

('Electric Waves,' p. 150), we should have at 12 metres distance from the oscillator $e^{p_1 ar} = \sqrt{e} = 1{\cdot}649$, a factor which can very considerably modify Q as a function of r, if t be not equal to ar, *i.e.*, if the disturbance have not just reached that point of the field.

Expressing Q in terms of the wave-length λ, and the period 2τ, we have

$$\frac{Q}{2\pi El/\lambda} = \sin^2\theta e^{-\nu(t/2\tau - r/\lambda)} \left\{ \frac{\cos\left[2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right]}{\cos\chi} + \frac{\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{2\pi r/\lambda} \right\} \quad . \quad \text{(ix)},$$

where $\nu = 2\pi p_1/p_2 = p_1 \times 2\tau$.

Thus we may write

$$\frac{Q}{2\pi El/\lambda} = \phi\left(\theta, \frac{r}{\lambda}, \frac{t}{\tau}\right),$$

where ϕ is a known function.

The next stage was to plot for a reasonable value of ν, the curves $\phi =$ constant for a series of values of Q at different intervals of time. In order to show the decadence of the strength of the field in the neighbourhood of the oscillator, t was given the 56 successive values $\frac{1}{4}\tau$, $\frac{2}{4}\tau$, $\frac{3}{4}\tau$, τ, $\frac{5}{4}\tau$ . . . $13\frac{3}{4}\tau$, 14τ; or the field-changes were traced for seven complete oscillations, at intervals of $\frac{1}{8}$ oscillation. This was done for a sphere of $1\frac{1}{4}$ wave-lengths round the oscillator, or taking the wave-length to be 9·6 metres, for a sphere of 12 metres radius round the centre of the oscillator.*

Values of $Q\lambda/(2\pi El)$ were chosen so as to give eight systems of curves with relative intensities

50, 30, 10, 1, − 1, − 10, − 30, − 50.

In the accompanying diagrams (Plates 1–7) the fine continuous curves† give the ± 50 intensity, the fine dotted curves the ± 30, the heavy continuous curves ± 10, and the heavy dotted curves ± 1. The outermost circle in each case is the boundary of the field explored; the innermost circle corresponds to the boundary of the area round the oscillator, within which it would hardly be legitimate to consider the oscillator a double electric point. The remaining circles are those for which Q = 0, and which separate positive from negative portions of the field. In the spaces between these circles the field must be considered to have alternative positive and negative intensity. It must of course be borne in mind that the curves only give a meridian section of the surfaces of equal intensity.

The work of calculating, plotting, and drawing such a long series of curves was

* HERTZ's diagrams only extend for three-quarters of a wave-length round the oscillator. Our diagrams suffice in extent to show the detachment of the loops preparatory to their outward propagation.

† The actual selection of fine and heavy, continuous and dotted curves to represent the several intensities, is due to the engraver working on the photographs of the original coloured diagrams. For special aid in the preparation of the diagrams and in their reproduction by photography, we have to heartily thank Messrs. E. WRENN and A. WHEELER respectively.

laborious, but as no investigation had been made of the damping out of an electromagnetic field, the work seemed worth undertaking. Some remarks may be made on the methods by which the arithmetic was carried out.

The equation to the curve being written

$$\sin^2\theta = c \times \psi(r),$$

where $c = Q\lambda/(2\pi E l)$ and

$$\psi(r) = e^{\nu(t/2\tau - r/\lambda)} \left\{ \frac{\cos\left[2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) \chi\right]}{\cos\chi} + \frac{\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{2\pi r/\lambda} \right\}^{-1},$$

c was given the values $\frac{1}{100}$, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{1}{2}$, and the limiting values of r ascertained, for which θ was real. r was then given a series of values between these limits altering by small differences, and the values of $\sin\theta$ calculated; frequent values of r were taken at portions of the curve where it was found to be turning, and thus a close approximation found to its form. The original diagrams on a scale of a metre to the inch were formed by joining up the calculated points and painting in the curves so formed. These were afterwards reduced by photography.*

The dying out of a part of the field of a particular strength is well illustrated in the diagrams. Take the dying out of strengths greater than $\pm$ 50 represented by the fine continuous line. In fig. 25 we see the last sensible loop of the field containing greater strengths than this passing away. Up to fig. 33 we can still trace this portion of the field as a dot. But in fig. 29 the oscillator has ceased to give fresh centres even of this strength, and in fig. 34 they have passed away entirely. Figs. 30 to 41 give the shrinking up and final disappearance of parts of the field with a strength greater than $\pm$ 30. Figs. 53 to 56 show the outward passage of the last loop of the field with a strength greater than $\pm$ 10, and another ten diagrams would have sufficed to show no trace of strengths greater than $\pm$ 1.

(4.) We will now proceed to find the electric and magnetic forces from (vi).

First, the electric force, Z, in direction of the axis of the oscillator:

$$Z = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda}\left[\sin^2\theta\,\frac{\sin\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+2\chi\right\}}{\cos^2\chi} + (2\cos^2\theta - \sin^2\theta) \times \frac{\cos\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\chi\right\}}{(2\pi r/\lambda)\cos\chi} + (2\cos^2\theta - \sin^2\theta)\,\frac{\sin\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)\right\}}{(2\pi r/\lambda)^2}\right] \quad \text{(x)}.$$

Second, the electric force, R, perpendicular to the axis of the oscillator:

* The diagrams have lost very considerably in the process of photography and engraving. It may possibly be that some of the finer loops and dots will appear not at all, or at least unclearly, after the blocks have been somewhat used. We hope shortly to have kinematograph films of the original diagrams available.

$$R = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \sin\theta\cos\theta \left[-\frac{\sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + 2\chi\right\}}{\cos^2\chi} + \frac{3\cos\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right\}}{(2\pi r/\lambda)\cos\chi} + \frac{3\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{(2\pi r/\lambda)^2}\right] \quad \text{(xi)}.$$

Now let

$$\phi_1(r, t) = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \left[\frac{\sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + 2\chi\right\}}{\cos^2\chi} - \frac{3\cos\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right\}}{(2\pi r/\lambda)\cos\chi} - \frac{3\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{(2\pi r/\lambda)^2}\right] \quad \text{(xii)},$$

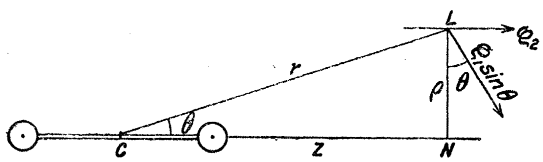
and

$$\phi_2(r, t) = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \left[\frac{2\cos\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right\}}{(2\pi r/\lambda)\cos\chi} + \frac{2\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{(2\pi r/\lambda)^2}\right] \quad \text{(xiii)}.$$

Then ϕ_1 and ϕ_2 are constant over any spherical surface about the centre of the oscillator at any time, and

$$\left.\begin{aligned} Z &= \phi_1 \sin^2\theta + \phi_2, \\ R &= -\phi_1 \sin\theta\cos\theta, \end{aligned}\right\} \quad \text{(xiv)}.$$

The electric force can thus be considered as compounded of a force ϕ_2 parallel to the axis of the oscillator, and depending only on the distance of the point of the field from its centre, and of a force $\phi_1 \sin\theta$, acting in the meridian plane perpendicular to the central distance of any point and towards the oscillator axis.



Clearly along the axis and in the equatorial plane $R = 0$, or the force is parallel to the axis, a result already deduced by HERTZ for a simple harmonic oscillation.* At very great distances we may neglect inverse squares and cubes of r, as compared with first powers, and accordingly ϕ_2 vanishes as compared with ϕ_1. In other words the electric force tends at great distances to become perpendicular to the radius C L, or the propagation to be *purely transverse.*

At a considerable distance from the origin we may take :

* 'Electric Waves,' pp. 142-3

$$Z = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu\,(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \sin^2\theta \, \frac{\sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + 2\chi\right\}}{\cos^2\chi},$$

$$R = -\frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu\,(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \sin\theta\cos\theta \, \frac{\sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + 2\chi\right\}}{\cos^2\chi} \quad . \quad . \quad \text{(xv)}.$$

The values in our notation obtained by HERTZ are:

$$Z = \frac{El\left(\frac{\pi}{\lambda}\right)^3}{2\pi r/\lambda} \sin^2\theta \sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right),$$

$$R = -\frac{El\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda} \sin\theta\cos\theta\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right).$$

Now it must be remembered that Q and R and Z are all zero until t is $> ar$ or $\frac{t}{2\tau} > \frac{r}{\lambda}$. Hence HERTZ's formulæ imply that at a considerable distance from the origin the intensity of the field increases *gradually* from zero. The formulæ (xv) appear, however, to show that the intensity at a distant point of the field rises abruptly from zero to the definite value:

$$\frac{El\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda} \sin\theta \, \frac{\sin 2\chi}{\cos^2\chi} = \frac{2El\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda} \sin\theta\tan\chi$$

as the wave reaches it.

The explanation of this is, however, that, while we make the oscillator start from zero charge, yet the initiation is sudden in so far as it requires definite initial values of the electric and magnetic forces. There is an impulsive action at the wave front due to the sudden starting of the oscillator, and the above expression only represents the electric force at a considerable distance from the oscillator, when the impulsive action of the wave front has just passed the point under consideration.

Turning now to the magnetic force P perpendicular to the meridian plane, we have by (vi):

$$P = a\,\frac{1}{\rho}\,\frac{dQ}{dt},$$

or,

$$P = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu\,(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \sin\theta\left[-\frac{\sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + 2\chi\right\}}{\cos^2\chi} + \frac{\cos\left\{2\pi\left(\frac{t}{\tau} - \frac{r}{\lambda}\right) + \chi\right\}}{(2\pi r/\lambda)\cos\chi}\right] \text{(xvi)},$$

$$= \phi_3 \sin\theta$$

where ϕ_3 is a function of r and t only, and is constant at any time for a spherical surface round the centre of the oscillator.

Clearly P does not, as in HERTZ's formula, appear to gradually rise from a zero value, but suddenly springs to the value

$$\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin\theta\left(\frac{1}{2\pi r/\lambda}-2\tan\chi\right),$$

or,

$$=-\frac{2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin\theta\tan\chi,$$

at a considerable distance from the oscillator.* This must again be interpreted as representing the value of the magnetic force immediately after the impulsive action at the wave front has passed by.

(5.) We shall now consider what modifications are made in the velocity of transmission owing to the damping of the wave train. HERTZ, GRAY, and others have considered this problem, but have confined their attention to the equatorial plane or the axis, and to a simple harmonic train. There appears to be no real simplicity gained by these limitations. Dealing first with magnetic force in (xvi), we may write the value of P above

$$\mathrm{P}=\mathrm{P}_0(r)\,e^{-\nu(t/2\tau-r/\lambda)}\sin\theta\cos\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0\right\}. \quad . \quad . \quad . \quad (\text{xvii})$$

where we have

$$\mathrm{P}_0(r)\cos\beta_0=\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\left(\frac{1}{2\pi r/\lambda}-2\tan\chi\right),$$

$$\mathrm{P}_0(r)\sin\beta_0=\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\left(\tan\chi\,\frac{1}{2\pi r/\lambda}+\frac{\cos 2\chi}{\cos^2\chi}\right).$$

Hence

$$\left.\begin{aligned}\mathrm{P}_0(r)&=\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{(2\pi r/\lambda)\cos\chi}\left\{1+\left(\frac{1}{2\pi r/\lambda}-\tan\chi\right)^2\right\}^{\frac{1}{2}},\\ \tan\beta_0&=\frac{\dfrac{\sin\chi\cos\chi}{2\pi r/\lambda}+\cos 2\chi}{\dfrac{\cos^2\chi}{2\pi r/\lambda}-\sin 2\chi}\end{aligned}\right\} \quad . \quad . \quad . \quad (\text{xviii}).$$

* The apparent equality of the initial electric and magnetic forces arises from the units selected by HERTZ, which have been here adopted for purposes of comparison between the two theories. See 'Electric Waves,' pp. 138–9.

From these we deduce

$$\cot(\beta_0 - \chi) = \frac{1}{2\pi r/\lambda} - \tan\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xix)},$$

$$P_0(r) = \frac{El(2\pi/\lambda)^3}{(2\pi r/\lambda)\cos\chi\sin(\beta_0 - \chi)} = \frac{El(2\pi/\lambda)^3\cos(\beta_0 - 2\chi)}{\cos^2\chi\sin^2(\beta_0 - \chi)} \quad . \quad . \quad . \quad \text{(xx)}.$$

From (xix)

$$\frac{1}{\sin^2(\beta_0 - \chi)}\frac{d\beta_0}{dr} = \frac{2\pi/\lambda}{(2\pi r/\lambda)^2},$$

or,

$$\frac{d\beta_0}{dr} = \frac{2\pi/\lambda}{(2\pi r/\lambda)^2}\frac{1}{\left(\frac{1}{2\pi r/\lambda} - \tan\chi\right)^2 + 1}.$$

To find the wave-speed, we have from (xvii) to find dr/dt from

$$2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_0 = \text{constant},$$

i.e.,

$$\frac{\lambda}{2\tau} - \frac{dr}{dt}\left(1 - \frac{\lambda}{2\pi}\frac{d\beta_0}{dr}\right) = 0.$$

Let $\lambda/2\tau = v$ as before, and $dr/dt = V_0$. Then

$$V_0/v = \frac{1}{1 - \frac{\xi}{1 + (\xi - \tan\chi)^2}},$$

where $\xi = \frac{1}{2\pi r/\lambda}$. Hence

$$\frac{V_0 - v}{v} = \frac{\xi^2}{1 + \tan^2\chi - 2\xi\tan\chi} = \frac{1}{2\tan\chi} \times \frac{\xi^2}{\operatorname{cosec} 2\chi - \xi} \quad . \quad . \quad . \quad . \quad \text{(xxi)}.$$

Now this result is quite independent of the *direction* of propagation, or the wave moves outwards with the same velocity at the same distance in all directions.

For $\chi = 0$, $\frac{V_0 - v}{v} = \xi^2$. This is the value given by GRAY for propagation in the equatorial plane,* but it is clearly independent of direction. At considerable distances ξ is very small, and therefore $V_0 = v$. Thus v is the limit of wave velocity as we recede from the source of disturbance.

Now a remarkable result flows from (xxi), which could not in any way be predicted from HERTZ's theory, neglecting the damping. HERTZ tells us that the velocity of propagation at the source is infinite, and GRAY draws the same conclusion, but (xxi) shows us that near the source the velocity of propagation, although very great, is

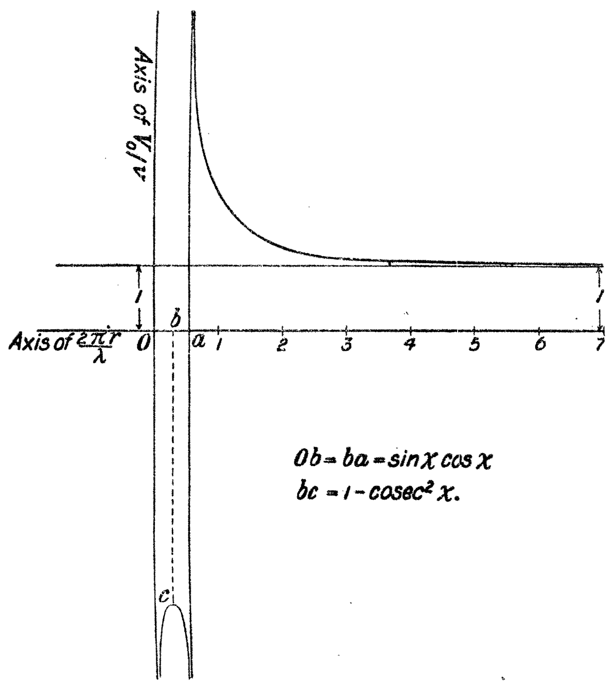
* 'Absolute Measurements,' vol. 2, p. 781.

negative. This holds until $\xi = \operatorname{cosec} 2\chi$, and then V_0 become indefinitely great positively. Thus, from the surface of the sphere $r = \frac{\lambda \sin 2\chi}{2\pi}$ the waves move inwards and outwards with indefinitely great velocities. When $\chi = 0$, this sphere closes in on the oscillator, and it will generally be well within the sphere round the oscillator within which it is not legitimate to apply the theory. But for a very rapidly damped wave train and a very considerable wave length, it is possible that its existence could be physically detected.

We may write (xxi)

$$\frac{V_0}{v} = 1 + \frac{\cos^2 \chi}{\frac{2\pi r}{\lambda}\left(\frac{2\pi r}{\lambda} - \sin 2\chi\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxii).}$$

Hence V_0/v is symmetrical about $2\pi r/\lambda = \frac{1}{2} \sin 2\chi$, and we have the following curve for V_0/v plotted to $2\pi r/\lambda$:



N.B.—This figure is purely diagrammatic. O*a* is not really comparable with O1, and *bc* is possibly 20,000 units of the vertical scale.

The diagram illustrates the rapidity with which the velocity of the magnetic disturbance approximates to v, and shows the minimum negative velocity $v \operatorname{cosec}^2 \chi$ occurring at a distance $r = \frac{\lambda}{4\pi} \sin 2\chi$ from the centre of the oscillator.

(6.) Turning now to the wave-speed for the electric force, we see from the remarks on (xiv) that we can treat the wave as made up of two components corresponding to

ϕ_1 and ϕ_2. This is really a kinematic resolution convenient for analysis of the wave phenomena. We might have resolved the displacement in other ways, possibly with equal advantage. But it must not be supposed that a purely kinematical resolution into wave components can have no physical significance. It is true that neither ϕ_1 nor ϕ_2 could exist by themselves, but this is practically true of all the wave resolutions the mathematical physicist habitually deals with. He is accustomed to consider in electro-magnetism waves of electric and magnetic force, neither have an independent existence; of radial and transverse electric displacement, one is impossible without the other. In the theory of the refraction and reflection of waves at the interface of elastic media, and its applications to the undulatory theory of light, we do not hesitate to separate the waves of transverse from longitudinal displacement and to speak of the former as having an independent existence, yet we are really separating kinematically what can only coexist. Lastly, consider, perhaps, the most familiar case, that of the longitudinal vibrations of rods; here the physicists carry the kinematic resolution so far that they often forget to mention the coexistence of the vibrations perpendicular to the axis of the rod, without which the longitudinal vibrations could not exist at all. The fact is that a purely kinematic resolution is often of first-class physical importance, for one or other of its factors admits of ready physical determination. Thus, in our present case, ϕ_2 is the sole component of electric force in the axis of the oscillator, and determined there it is determined for all points of space at the same distance from the centre. Again, $\phi_1 + \phi_2$ is the component of electric force in the equatorial plane, and with determinations there, ϕ_2 will be known at all points of space. But it was precisely in the direction of the axis and in the equatorial plane that HERTZ made his chief experiments. Thus there seems considerable advantage in reducing the analysis of the electric force to two functions, both independent of the latitudes and varying only with the distance from the oscillator, which can be directly tested in the localities HERTZ found best suited to experimental enquiries. ϕ_1 and ϕ_2 are both independent of the latitude, and give wave speeds V_1 and V_2 having the same values for all points at the same distances from the centre of the oscillator.

The reader will notice that in taking ϕ_2 and $\phi_1 \sin\theta$ as our constituents we have resolved the electric force into two components, one along the axis and one transverse to the ray, and these components will not generally be at right angles. Neither will represent the total force in the given direction; they are transverse and axial components and not total transverse and total axial electric forces.

The total force perpendicular to axis $= -\phi_1 \sin\theta\cos\theta$;
The total axial electric force . . $= \phi_1 \sin^2\theta + \phi_2$;
The total transverse electric force . $= -(\phi_1 + \phi_2)\sin\theta$; and
The total radial electric force . . $= \phi_2 \cos\theta$.

It will be observed that the amplitude of the axial component does not change with θ, but that the amplitude of both the total transverse and the total electric radial forces does. So far as the theory of wave transmission goes, the wave-speed of the total radial force is really discussed under the treatment of ϕ_2, our axial component, in Art. 7. Similarly, the total transverse electric force has a wave-speed determined from $\phi_1 + \phi_2$ and, therefore, it will be found fully discussed under our treatment of the function $\phi_1 + \phi_2$, the equatorial wave in Art. 12. In addition, our analysis enables us to deal separately with that portion of the total transverse force ϕ_1, or the transverse component in our case, which alone is propagated to considerable distances, and which gives at all distances the total force perpendicular to the axis.

(7.) Let us deal first with the case of the component force parallel to the axis or the ϕ_2 factor of the total radial electric force. We may write

$$\phi_2 = \mathrm{P}_2(r)\, e^{-\nu(t/2\tau - r/\lambda)} \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_2\right\} \quad . \quad . \quad . \quad . \quad \text{(xxiii)},$$

where

$$\mathrm{P}_2(r) \sin\beta_2 = 2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi^2$$

$$\mathrm{P}_2(r) \cos\beta_2 = -\, 2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi^2 (\tan\chi - \xi).$$

Thus,

$$\mathrm{P}_2(r) = 2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi^2 \{1 + (\tan\chi - \xi)^2\}^{\frac{1}{2}},$$

and,

$$\cot\beta_2 = -\tan\chi + \xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxiv)}.$$

Hence

$$\frac{1}{\sin^2\beta_2}\frac{d\beta_2}{dr} = \frac{\lambda}{2\pi}\frac{1}{r^2} = \frac{2\pi}{\lambda}\xi^2,$$

and differentiating $2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_2 = 0$ with regard to the time to find $\mathrm{V}_2 = dr/dt$, we have

$$v = \mathrm{V}_2(1 - \xi^2\sin^2\beta_2).$$

Thus the wave is

$$\phi_2 = \frac{2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{(2\pi r/\lambda)^2}\left\{1 + \left(\tan\chi - \frac{\lambda}{2\pi r}\right)^2\right\}^{\frac{1}{2}} e^{-\nu(t/2\tau - r/\lambda)} \times \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_2\right\} \quad \text{(xxv)},$$

and its velocity is given by

$$\mathrm{V}_2/v = 1 + \frac{\cos^2\chi}{\frac{2\pi r}{\lambda}\left(\frac{2\pi r}{\lambda} - \sin 2\chi\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxvi)}.$$

We see at once that $\mathrm{V}_2 = \mathrm{V}_0$, or the magnetic wave and the wave of component electric force parallel to the axis are propagated everywhere with the same velocity at the same distance from the axis. At the same time the amplitude of this electric wave varies inversely as the *square* of the distance from the centre of the oscillator for considerable distances, while the amplitude of the magnetic wave varies under the same conditions inversely as the distance. Thus the effect of the former is rapidly insensible as compared with the latter. Meanwhile it is important to observe how this wave keeps pace with the magnetic wave.

(8.) We may now consider ϕ_1, which gives the transverse electric component $\phi_1 \sin\theta$ and the total electric force perpendicular to the axis, *i.e.*, $-\phi_1 \sin\theta \cos\theta$.*

$$\phi_1 \sin\theta = \mathrm{P}_1(r)\, e^{-\nu(t/2\tau - r/\lambda)} \sin\theta \times \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_1\right\} \quad . \quad \text{(xxvii)},$$

where

$$\mathrm{P}_1(r)\cos\beta_1 = \mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi\,(1 - \tan^2\chi + 3\xi\tan\chi - 3\xi^2),$$

$$\mathrm{P}_1(r)\sin\beta_1 = \mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi\,(2\tan\chi - 3\xi).$$

Hence

$$\mathrm{P}_1(r) = \frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi}{\cos^2\chi}\{1 - 3\xi\sin 2\chi + 3\xi^2\cos^2\chi\,(1 + 4\cos^2\chi) - 3\xi^3\sin 2\chi\cos^2\chi + 9\xi^4\cos^4\chi\} \quad . \quad . \quad \text{(xxviii)},$$

and

$$\tan\beta_1 = \frac{2\tan\chi - 3\xi}{1 - \tan^2\chi + 3\xi\tan\chi - 3\xi^2}.$$

From the last equation we deduce, if $\zeta = 1/\xi = \dfrac{2\pi r}{\lambda}$,

$$\tan(\beta_1 - 2\chi) = \frac{-3(1 + \tan^2\chi)(\zeta - \sin 2\chi)}{\zeta^2(1 + \tan^2\chi)^2 - 3\zeta\tan\chi\,(1 + \tan^2\chi) - 3(1 - \tan^2\chi)} = \frac{-3\epsilon}{(\epsilon + \gamma)^2 - 3(1 - \gamma^2)}$$

where

$$\epsilon = (1 + \tan^2\chi)(\zeta - \sin 2\chi), \qquad \gamma = \tfrac{1}{2}\tan\chi.$$

Hence

$$\frac{1}{\cos^2(\beta_1 - 2\chi)}\frac{d\beta_1}{dr} = 3(1 + \tan^2\chi)\frac{2\pi}{\lambda} \times \frac{3 + \epsilon^2 - 4\gamma^2}{\{(\epsilon + \gamma)^2 - 3(1 - \gamma^2)\}^2},$$

$$\frac{d\beta_1}{dr} = \frac{2\pi}{\lambda}\,\frac{3(1 + 4\gamma^2)(3 + \epsilon^2 - 4\gamma^2)}{(\epsilon^2 + 2\gamma\epsilon + 4\gamma^2 - 3)^2 + 9\epsilon^2}.$$

* This again is an important physical significance for ϕ_1, and enables it to be readily differentiated physically from ϕ_2 and $\phi_1 + \phi_2$.

We must now differentiate $\frac{2\pi}{\lambda}(vt - r) + \beta_1 = 0$, to get the velocity V_1, and we find

$$v = V_1 \left(1 - \frac{\lambda}{2\pi}\frac{d\beta_1}{dr}\right),$$

or,

$$\frac{V_1}{v} = 1 + \frac{1}{\frac{2\pi}{\lambda}\frac{1}{d\beta/dr} - 1}.$$

Substituting for $d\beta_1/dr$, we have, after reductions :

$$\frac{V_1}{v} = 1 + \frac{3(1 + 4\gamma^2)\{\epsilon^2 - (4\gamma^2 - 3)\}}{\epsilon^4 + 4\gamma\epsilon^3 + 4\gamma(4\gamma^2 - 3)\epsilon + 16\gamma^2(4\gamma^2 - 3)}.$$

Now let

$$\left.\begin{array}{l}\zeta_0 = \sin 2\chi \\ \zeta_1^2 = \cos^2\chi(3\cos^2\chi - \sin^2\chi)\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxix}).$$

Then

$$\zeta_0 = \frac{4\gamma}{1 + 4\gamma^2}, \qquad \zeta_1^2 = \frac{3 - 4\gamma^2}{(1 + 4\gamma^2)^2},$$

and

$$\frac{V_1}{v} = 1 + \frac{3\cos^2\chi\{(\zeta - \zeta_0)^2 + \zeta_1^2\}}{(\zeta - \zeta_0)^4 + \zeta_0(\zeta - \zeta_0)^3 - \zeta_0\zeta_1^2(\zeta - \zeta_0) - \zeta_0^2\zeta_1^2}$$

$$= 1 + 3\cos^2\chi\frac{\{(\zeta - \zeta_0)^2 + \zeta_1^2\}}{\zeta\{(\zeta - \zeta_0)^3 - \zeta_0\zeta_1^2\}}.$$

Let $\zeta_0 = 2\pi r_0/\lambda$, $\zeta_1 = 2\pi r_1/\lambda$. Then

$$\frac{V_1}{v} = 1 + \frac{3\cos^2\chi}{(2\pi/\lambda)^2}\frac{(r - r_0)^2 + r_1^2}{r\{(r - r_0)^3 - r_0 r_1^2\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxx}).$$

This gives the velocity V_1 at each distance r from the origin of the transverse electric wave. Its amplitude is given by (xxviii).

When r is great, the amplitude approaches the value

$$\frac{El(2\pi/\lambda)^3}{\cos^2\chi(2\pi r/\lambda)}.$$

Therefore at such distances

$$\phi_2/\phi_1 = 2(\lambda/2\pi r),$$

or ϕ_2 is insensible as compared with ϕ_1. Even at distances 5 to 10 times the wave length, ϕ_2 will be very small as compared with ϕ_1; that is to say, the electric vibrations at comparatively short distances are to all intents and purposes *transverse*.

Turning now to the velocity V_1, we will first endeavour to trace its changes. Let us write $r_0/r_1 = q^3$, where q is generally a small quantity. Then the denominator of the expression for $V_1 - v$ in terms of r (see xxx) may be written

$$r(r - r_0 - qr_1)\{(r - r_0)^2 + (r - r_0)qr_1 + q^2r_1^2\}.$$

The second factor is negative for $r < r_0 + qr_1$, then vanishes for $r = r_0 + qr_1$, and is ever afterwards positive.

The third factor would vanish for imaginary values only, and since it is positive for $r = r_0$, it remains positive always. This supposes that r_1^2 is positive. Similarly the numerator in the value of ϕ_1, or $(r - r_0)^2 + r_1^2$ will always be positive, if r_1^2 be positive. Now by (xxix), r_1^2 cannot be negative unless $\tan\chi > \sqrt{3}$ or $\chi > 60°$. This corresponds to a degree of damping in which the amplitude would be reduced to ·000019 of itself every period, or a degree immensely higher than that of the usual Hertzian oscillator. Hence both the numerator and the third factor of the denominator are always positive.

Accordingly $V_1 - v$ is negatively infinite for $r = 0$, becomes negatively finite, but very large, until $r = r_0 + qr_1$ when it again becomes negatively infinite, after this it becomes positively infinite, and rapidly decreases to zero as r increases. Thus, as in the case of the other waves, there is a sphere round the oscillator within which the waves move inwards for transverse electric vibrations. This sphere is of somewhat larger radius ($r_0 + qr_1$ as compared with r_0) than in the case of the magnetic wave. In terms of χ the radius of this sphere is given by

$$\frac{\lambda}{2\pi}(\sin 2\chi)\{(\tfrac{3}{4}\operatorname{cosec}^2\chi - 1)^{1/3} + 1\}.$$

For example, when $\tan\chi = \cdot 2/\pi$ or $\chi = 3° 38' 33''\cdot 5$, we find for the radius of this sphere, 6·69604 r_0, or between six and seven times the radius of the sphere within which the velocity of the magnetic wave is negative. Substituting the value of r_0, the radius = ·135λ. Hence, for a small oscillator, say $\frac{1}{10}$ of a wave-length long, this sphere would be well outside the sphere ·05 of a wave-length circumscribing the oscillator, and thus practically within the field to which our theory might be approximately applied. The inward moving wave of transverse electric vibrations ought thus to be capable of experimental demonstration.

We have, in the next place, to find the minimum negative velocity, and its distance from the centre.

Writing $r - r_0 = u$ and $r_0 = q^3r_1$, we have to find the maxima or minima of

$$\frac{u^2 + r_1^2}{(u + q^3r_1)(u^3 - q^3r_1^3)}.$$

Differentiating, we find for the required values

$$2\left(\frac{u}{r_1}\right)^5 + q^3\left(\frac{u}{r_1}\right)^4 + 4\left(\frac{u}{r_1}\right)^3 + 4q^3\left(\frac{u}{r_1}\right)^2 + 2q^6\left(\frac{u}{r_1}\right) - q^3 = 0 \quad . \quad \text{(xxxi)}.$$

Now, at $r = r_0$ it is easy to see that the curve in which $\frac{V_1 - v}{v}$ is plotted to r is still sloping towards the horizontal. For if $\frac{V_1 - v}{v} = \eta$ and $r - r_0 = \xi$, its equation is approximately

$$\eta = -\frac{3\cos^2\chi}{(2\pi/\lambda)^2}\frac{1}{r_0(r_0+\xi)},$$

or, η decreases with increase of ξ hyperbolically.

Hence the minimum negative value sought must lie between r_0 and $r_0 + qr_1$, say at $r_0 + \eta q r_1$, where η is less than unity but not necessarily very small. Thus we must put $u/r_1 = \eta q_1$ where q is small, and endeavour to solve (xxxi). Substituting, we have

$$2q^5\eta^5 + q^7\eta^4 + 4q^3\eta^3 + 4q^5\eta^2 + 2q^7\eta - q^3 = 0,$$

or,

$$2q^2\eta^5 + q^4\eta^4 + 4\eta^3 + 4q^2\eta^2 + 2q^4\eta - 1 = 0.$$

Clearly, η must be of the form $\eta_0 + \eta_1 q^2 + \eta_2 q^4 + \eta_3 q^6 + \&c.$ Let us substitute this value and equate the successive powers of q^2 as well as the constant term to zero. We find

$$4\eta_0^3 - 1 = 0,$$
$$\eta_0^5 + 6\eta_1\eta_0^2 + 2\eta_0^2 = 0,$$
$$10\eta_0^4\eta_1 + \eta_0^4 + 12\eta_0^2\eta_2 + 12\eta_0\eta_1^2 + 8\eta_0\eta_1 + 2\eta_0 = 0,$$
$$10\eta_0^4\eta_2 + 20\eta_0^3\eta_1^2 + 4\eta_0^3\eta_1 + 12\eta_0^2\eta_3 + 24\eta_0\eta_1\eta_2 + 4\eta_1^3 + 8\eta_0\eta_2 + 4\eta_1^2 + 2\eta_1 = 0.$$

These equations give us

$$\left.\begin{aligned}\eta_0 &= (\tfrac{1}{4})^{1/3} = \cdot 62996\\ \eta_1 &= -\tfrac{3}{8} = -\cdot 375\\ \eta_2 &= 0\\ \eta_3 &= \cdot 01476\end{aligned}\right\} \quad \ldots \ldots \quad \text{(xxxii).}$$

Thus

$$\eta = \cdot 62996 - \cdot 375q^2 + \cdot 01476q^6.$$

We find at once that the required radius of the sphere at which the velocity of the electric transverse wave takes a minimum negative value

$$= r_0 + \eta q r_1 = r_0(1 + \eta/q^2)$$
$$= r_0\left(\frac{\cdot 62996}{q_2} + \cdot 625 + \cdot 01476q^4\right). \quad \ldots \ldots \quad \text{(xxxiii).}$$

The next term in the expression for the radius would involve q^6 and may generally be neglected. If the point of minimum velocity were half-way between the two points of infinite negative velocity, we should have the radius $= \frac{1}{2}(r_0 + qr_1)$ $= r_0\left(\frac{\cdot 5}{q^2} + \cdot 5\right)$. Hence, the point of minimum velocity is more than half-way towards the outer point of infinite velocity. For the particular oscillator referred to on p. 162, $q^2 = \cdot 17556$, and the radius of the sphere of minimum velocity $= 4\cdot 21\, r_0$ nearly, while the sphere of infinite negative velocity has $6\cdot 70\, r_0$ for radius nearly (see p. 174).

At a distance from the centre of the oscillator we have very approximately

$$\frac{V_1 - v}{v} = \frac{3\cos^2\chi}{(2\pi r/\lambda)^2},$$

or,

$$V_1 - v = 3(V_2 - v).$$

Thus at some distance from the centre of the oscillator, the excess of the velocity of this component transverse electric wave (or of the total electric wave perpendicular to the axis) over the velocity of light is three times the excess of the velocity of the magnetic wave.

In order to find the distance from the centre at which this component transverse electric and the magnetic waves have equal negative velocities we must make

$$\frac{1}{r(r - \frac{3}{2}r_0)} = \frac{3\{(r - r_0)^2 + r_1^2\}}{r\{(r - r_0)^3 - r_0 r_1^2\}},$$

or, putting $r - r_0 = u$, solve the cubic

$$2u^3 + 3r_1^2 u + r_0 r_1^2 = 0.$$

Take $u = -\eta r_0 = -\eta q^3 r_1$; hence

$$2\eta^3 q^6 + 3\eta - 1 = 0.$$

Thus,

$$\eta = \tfrac{1}{3} - \tfrac{2}{81} q^6, \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxiv)}.$$

We conclude accordingly that the two velocities are negatively equal at a distance from the centre equal to $\frac{2}{3}r_0(1 + \frac{1}{27}q^6) = \frac{2}{3}r_0$, nearly.

This equal negative velocity is $v\left(1 - \frac{1\cdot 125}{\sin^2\chi}\right)$, nearly. Similarly, the minimum negative velocity of the electric transverse wave may be shown to be

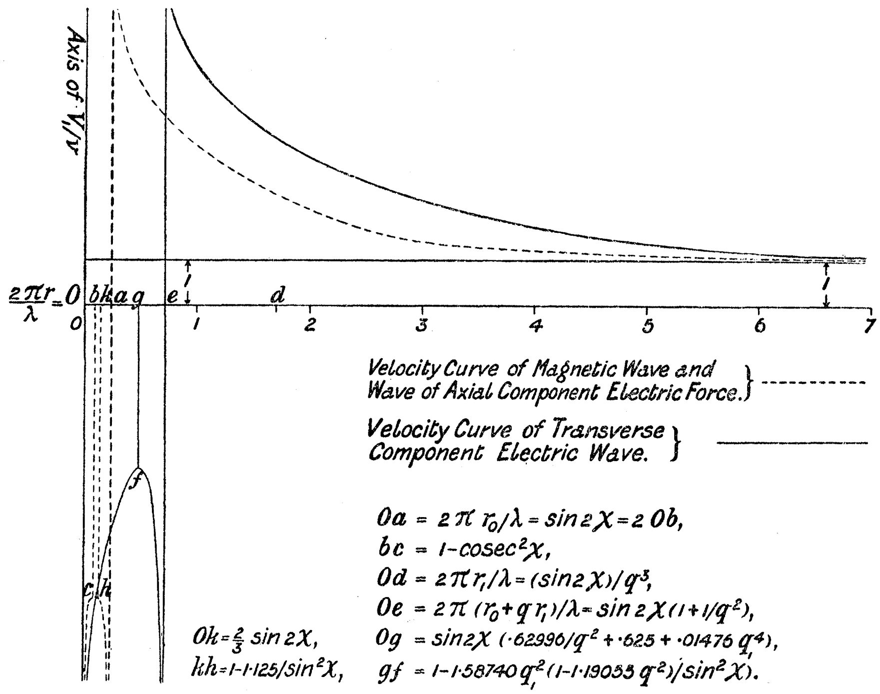
$$v\left\{1 - \frac{1\cdot 58740 q^2}{\sin^2\chi}(1 - 1\cdot 19055 q^2 + 2\cdot 12601 q^4)\right\} \quad . \quad . \quad . \quad . \quad \text{(xxxv)},$$

where $q^2 = \dfrac{1}{\{\frac{3}{4}\operatorname{cosec}^2\chi - 1\}^{1/3}}$ as before.

For the special oscillator discussed above it equals $-59\cdot 545$, or is slightly less than 60 times the velocity of light. The minimum negative velocity of the transverse magnetic wave $= -248 \cdot 46v$, and the equal negative velocity of both waves $= -279 \cdot 64v$. Thus again it appears more feasible to test experimentally the negative velocity of this transverse electric wave than that of the magnetic wave.

With regard to these negative waves, we can only cite what HERTZ has remarked on the infinite motion which occurs with a steady oscillator ('Electric Waves,' p. 146): "At an infinitesimal distance from the origin the velocity of propagation is even infinite. This is the phenomenon which, according to the old mode of expression, is represented by the statement that upon the electromagnetic action, which travels with velocity $^1/A$ [our v], there is superposed an electrostatic force travelling with infinite velocity. In the sense of our theory we more correctly represent the phenomenon by saying that fundamentally the waves which are being developed do not owe their formation solely to processes at the origin, but arise out of the conditions of the whole of the surrounding space, which latter, according to our theory, is the true seat of the energy."

The following figure gives the velocity curve of the transverse component electric wave or of the total electric wave perpendicular to the axis *diagrammatically.*



(9.) It is noteworthy that HERTZ, speaking of the electric force in the equatorial plane, *i.e.*,

$$Z = \phi_1 + \phi_2,$$

says that it cannot *possibly* be broken up into two simple waves travelling with different velocities ('Electric Waves,' p. 151). The explanation of HERTZ's difficulty is, perhaps, simple; for, having put $\theta = \pi/2$, the distinction between ϕ_1 and ϕ_2 was lost in his theory. We may consider these to be two separate waves travelling in the equatorial plane, both indeed transverse just for this plane, but one ϕ_1 travelling with the velocity of the transverse component electric wave and the other with the velocity of the magnetic wave, or that of the wave of axial component electric force. Thus HERTZ's original explanation of the irregularity of the interferences which did not succeed each other at equal distances, but with more rapid changes in the neighbourhood of the oscillator, seems justified by the fuller theory. Namely, he explained this behaviour by the supposition "that the total force might be split up into two parts, of which the one, the electromagnetic, was propagated with the velocity of light, while the other, the electrostatic, was propagated with greater and perhaps infinite velocity." Actually we may resolve into two waves; the transverse component electric wave is propagated with greater velocity than the wave of axial component electric force, which has the velocity of the magnetic wave. Many of HERTZ's experiments were made at a comparatively short distance from the oscillator, and some of the discrepancies he noted between his theory and experiment seem capable of explanation by aid of the velocity diagram given above.

(10.) There is another quantity which HERTZ considers at length, namely, the rate of change of the magnetic force, which gives the integral force of induction round a small circle in a plane perpendicular to the magnetic force. We shall now accordingly consider the expression for $d\mathrm{P}/dt$ from (xvii):

$$\begin{aligned}\frac{d\mathrm{P}}{dt} &= \mathrm{P}_0(r)\sin\theta\, e^{-\nu(t/2\tau - r/\lambda)}\left\{-p_1\cos\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0\right)\right.\\ &\qquad\left.-p_2\sin\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0\right)\right\}\\ &= -\frac{\mathrm{P}_0(r)\,v\,(2\pi/\lambda)\sin\theta}{\cos\chi}\, e^{-\nu(t/2\tau - r/\lambda)}\sin\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0+\chi\right)\\ &= -\frac{\mathrm{P}_0(r)\,v\,(2\pi/\lambda)\sin\theta}{\cos\chi}\, e^{-\nu(t/2\tau - r/\lambda)}\sin\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_3\right) \quad . \quad \text{(xxxvi)},\end{aligned}$$

where $\beta_3 = \beta_0 + \chi$, and therefore from (xix),

$$\cot(\beta_3 - 2\chi) = \frac{1}{2\pi r/\lambda} - \tan\chi.$$

A more convenient form is easily deduced, namely:

$$\tan\left(\beta_3 - 3\chi\right) = \frac{2\pi}{\lambda}\left(r - \tfrac{1}{2}r_0\right)\Big/\cos^2\chi \quad . \quad . \quad . \quad . \quad \text{(xxxvii)},$$

where $r_0 = \dfrac{\lambda}{2\pi}\sin\chi$ as before.

Now let δ_3 give the phase of the sine term in (xxxvi), or :

$$\delta_3 = \frac{2\pi r}{\lambda} - \beta_3.$$

Comparing with (xxiv), we see $\beta_3 = \beta_2 + 2\chi$, or :

$$\delta_3 = \delta_2 - 2\chi.$$

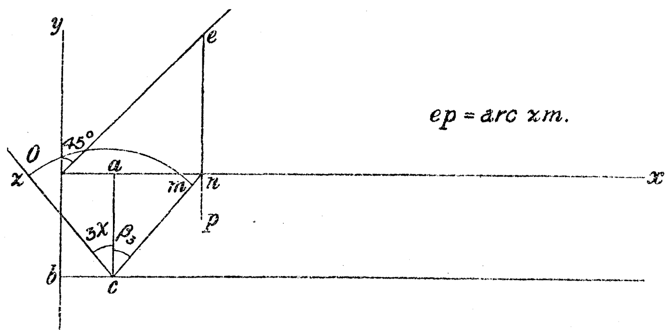
Thus the phase of the wave of magnetic induction always differs by a constant from that of the wave of axial component electric force.

Since $d\beta_3/dr = d\beta_2/dr$, the velocity of the wave of magnetic induction equals that of the wave of axial component electric force, and is given by the V_2 of (xxvi), or the diagram of p. 169.

(11.) In our diagram (p. 184) we have plotted δ_3 from the calculated curve for δ_2, but it is interesting to note the following graphical construction for finding it directly. Let $\zeta = 2\pi r/\lambda$ as before.

Let Ox be the axis of ζ, Oy of δ_3, then $\delta_3 = \zeta - \beta_3$. Take O$b = \cos^2\chi$, O$a = \frac{1}{2}(2\pi r_0/\lambda) = \frac{1}{2}\zeta_0$, and let the vertical through a meet the horizontal through b in c. Let On equal ζ, then $an = \zeta - \frac{1}{2}\zeta_0$ and $an/ca = (\zeta - \frac{1}{2}\zeta_0)/\cos^2\chi = \tan(\beta_3 - 3\chi)$, or the $\angle\, acn = \beta_3 - 3\chi$.

Take cz, so that $\angle\, zca = 3\chi$, then $\beta_3 = \angle\, zcn$. If a circle be described round c with radius $cz = 1$, the arc zm corresponding to the angle zcn is the required length β_3. Subtract this arc (rectified, say, by RANKINE'S rule) from the corresponding ordinate ne of the 45° line through the origin ($\delta = \zeta$), and we find the true value of $\delta_3 = np$. This construction was actually gone through and compared with the calculated values for verification.



N.B.—The angle χ has been much exaggerated for diagrammatic purposes.

(12.) Before we refer the reader to our diagram giving the phases of the various waves, and describe how they were obtained, it seems well to consider the combined wave of total transverse electrical force in the equatorial plane. The resultant electrical force in this plane may be considered, as we have already seen, to consist of two parts, having different wave-speeds; but, possibly, if a receiver were sufficiently small, it might not be possible to differentiate them, and HERTZ'S theoretical view of their unity seems finally to have mastered his experimental suspicion as to the possible coexistence of two waves in this plane (see our p. 177). Accordingly we have worked out both the velocity and phase of this compound wave, which at the same time determines the total transverse wave (see p. 170), in order that a comparison may be made with HERTZ's results for an undamped single wave in the equatorial plane.

Returning to the formula (x) and putting $\theta = \frac{1}{2}\pi$, it may be read as:

$$Z = Z_0(r)\, e^{-\nu(t/2\tau - r/\lambda)} \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_4\right\} \quad . \quad . \quad . \quad . \quad \text{(xxxviii)},$$

where if $\xi = \dfrac{1}{2\pi r/\lambda}$:

$$Z_0(r) \sin\beta_4 = (2\tan\chi - \xi)\, \mathrm{E}l\,(2\pi/\lambda)^3\, \xi,$$

$$Z_0(r) \cos\beta_4 = (1 - \tan^2\chi + \xi\tan\chi - \xi^2)\, \mathrm{E}l\,(2\pi/\lambda)^3\, \xi.$$

Hence,

$$\tan\beta_4 = (2\tan\chi - \xi)\Big/(1 - \tan^2\chi + \xi\tan\chi - \xi^2) \quad . \quad . \quad . \quad \text{(xxxix)}.$$

From this we deduce:

$$\tan(\beta_4 - 2\chi) = \frac{-\cos^2\chi\left(\dfrac{2\pi r}{\lambda} - \sin 2\chi\right)}{\left(\dfrac{2\pi r}{\lambda} - \frac{1}{4}\sin 2\chi\right)^2 - \cos^2\chi\,(\cos^2\chi - \frac{3}{4}\sin^2\chi)}$$

$$= \frac{-c_0(\zeta - \zeta_0)}{(\zeta - \frac{1}{4}\zeta_0)^2 - c_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xl)},$$

where

$$c_0 = \cos^2\chi, \qquad \zeta_0 = \sin 2\chi, \qquad c_1^2 = \cos^2\chi\,(\cos^2\chi - \tfrac{3}{4}\sin^2\chi).$$

This formula is fairly well adapted for logarithmic calculation. The phase may be obtained by taking

$$\delta_4 = \zeta - \beta_4.$$

It may be compared with the value for δ_1, obtained from β_1, on p. 172, where we easily find

$$\tan(\beta_1 - 2\chi) = \frac{-3\cos^2\chi\left(\dfrac{2\pi r}{\lambda} - \sin 2\chi\right)}{\left(\dfrac{2\pi r}{\lambda} - \frac{3}{4}\sin 2\chi\right)^2 - 3\cos^2\chi\,(\cos^2\chi - \frac{1}{4}\sin^2\chi)}$$

$$= \frac{-3c_0(\zeta - \zeta_0)}{(\zeta - \frac{3}{4}\zeta_0)^2 - c_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xli)},$$

if

$$c_2^2 = 3\cos^2\chi\,(\cos^2\chi - \tfrac{1}{4}\sin^2\chi).$$

As before

$$\delta_1 = \zeta - \beta_1.$$

This again readily lends itself to logarithmic calculation.

Both have for asymptote the same 45° line, namely,

$$\delta = \zeta - (\pi + 2\chi).$$

We now proceed to trace the velocity of the electrical disturbance in the equatorial plane.

We have

$$\tan \beta_4 = \frac{2 \tan \chi - \xi}{1 - \tan^2 \chi + \xi \tan \chi - \xi^2},$$

whence, by differentiation, we find

$$\frac{\lambda}{2\pi} \frac{d\beta_4}{dr} = \frac{\xi^2 \{1 + \tan^2 \chi - 4\xi \tan \chi + \xi^2\}}{(1 + \tan^2 \chi)^2 - 2\xi \tan \chi (1 + \tan^2 \chi) - \xi^2 (1 - 3 \tan^2 \chi) - 2\xi^3 \tan \chi + \xi^4}.$$

Hence, if V_4 be the velocity and $v = \lambda/(2\tau)$, as before

$$\frac{V_4}{v} = 1 + \frac{(\lambda/2\pi).(d\beta_4/dr)}{1 - (\lambda/2\pi).(d\beta_4/dr)}$$

$$= 1 + \frac{\xi^2 (1 + \tan^2 \chi - 4\xi \tan \chi + \xi^2)}{(1 + \tan^2 \chi)^2 - 2\xi \tan \chi (1 + \tan^2 \chi) - 2\xi^2 (1 - \tan^2 \chi) + 2\xi^3 \tan \chi}.$$

Now put

$$\zeta = 1/\xi = 2\pi r/\lambda,$$

$$\frac{V_4}{v} = 1 + \frac{\cos^2 \chi \left\{\left(\zeta - \frac{2 \tan \chi}{1 + \tan^2 \chi}\right)^2 + \frac{1 - 3 \tan^2 \chi}{(1 + \tan^2 \chi)^2}\right\}}{\zeta \left\{\zeta^3 - \zeta^2 \frac{2 \tan^2 \chi}{1 + \tan^2 \chi} - 2\zeta \frac{1 - \tan^2 \chi}{(1 + \tan^2 \chi)^2} + \frac{2 \tan \chi}{(1 + \tan^2 \chi)^2}\right\}}.$$

Again put

$$\zeta_0 = \frac{2 \tan \chi}{1 + \tan^2 \chi} = \sin^2 \chi = \frac{2\pi}{\lambda} r_0,$$

$$\zeta_2^2 = \frac{1 - 3 \tan^2 \chi}{(1 + \tan^2 \chi)^2} = \cos^2 \chi (\cos^2 \chi - 3 \sin^2 \chi) = \frac{4\pi^2}{\lambda^2} r_2^2.$$

Then

$$\zeta_0^2 + \zeta_2^2 = \cos^2 \chi, \qquad \zeta_0 (\zeta_0^2 + \zeta_2^2) = 2 \tan \chi \big/ (1 + \tan^2 \chi)^2,$$

$$\zeta_2^2 + \tfrac{1}{2} \zeta_0^2 = \cos^2 \chi (\cos^2 \chi - \sin^2 \chi) = (1 - \tan^2 \chi) \big/ (1 + \tan^2 \chi)^2.$$

Thus we may write the above result

$$\frac{V_4}{v} = 1 + \frac{\cos^2 \chi \{(\zeta - \zeta_0)^2 + \zeta_2^2\}}{\zeta \{\zeta^3 - \zeta_0 \zeta^2 - (\zeta_0^2 + 2\zeta_2^2) \zeta + \zeta_0 (\zeta_0^2 + \zeta_2^2)\}}$$

$$= 1 + \frac{\cos^2 \chi \{(\zeta - \zeta_0)^2 + \zeta_2^2\}}{\zeta \{(\zeta - \zeta_0)^2 (\zeta + \zeta_0) - 2\zeta_2^2 (\zeta - \zeta_0) - \zeta_2^2 \zeta_0\}}$$

$$= 1 + \frac{\cos^2 \chi \{(r - r_0)^2 + r_2^2\}}{(2\pi/\lambda)^2 r \{(r - r_0)^2 (r + r_0) - 2r_2^2 (r - r_0) - r_2^2 r_0\}} \quad . \quad . \quad . \quad \text{(xlii)}.$$

This is a form directly comparable with (xxx).

If the wave be not damped, $\zeta_0 = 0$, $\zeta_2 = 1$, which gives

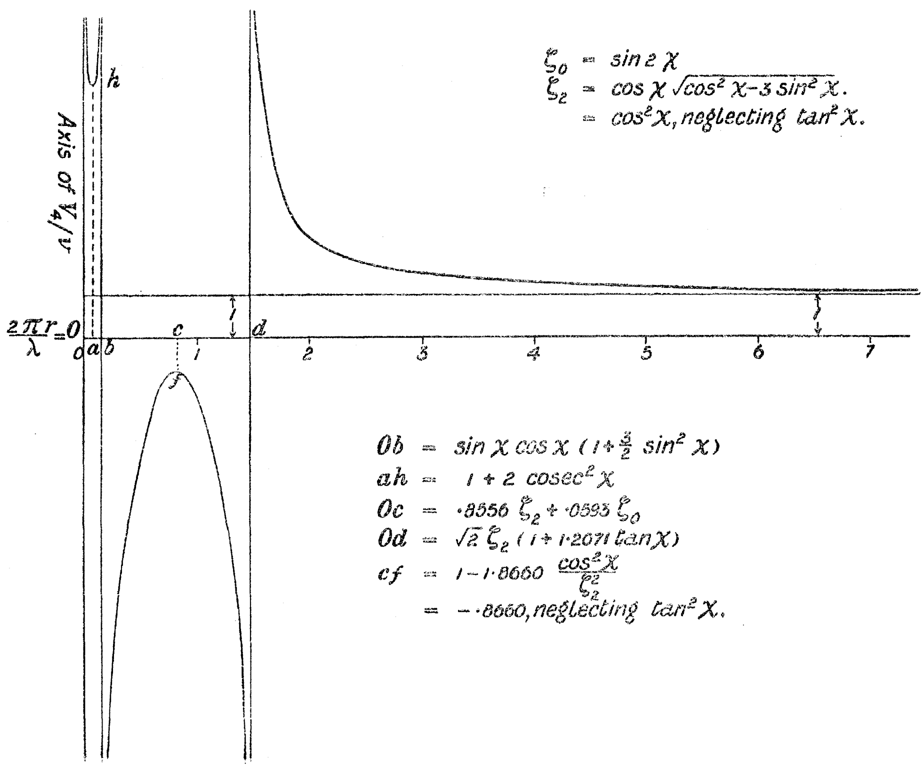
$$\frac{V_4}{v} = 1 + \frac{1 + \zeta^2}{\zeta (\zeta^3 - 2\zeta)} = \frac{1 - \zeta^2 + \zeta^4}{\zeta^2 (\zeta^2 - 2)},$$

a value identical with that given by GRAY.*

* 'Absolute Measurements,' vol. 2, Part II.,

The curve giving the velocity of this wave is found to be even more complex than that for either the wave of transverse or of axial component electric force alone (p. 177). In the neighbourhood of the origin the wave starts with infinite positive velocity; this falls to a finite but very great positive velocity at about $\frac{1}{4}\zeta_0$, and then rises again and becomes infinite at about $\frac{1}{2}\zeta_0$. Beyond this we have a region of negative velocity, starting with an infinite negative velocity, and falling to a velocity even less than that of light at about $\cdot 8556\,\zeta_2$. Then the velocity increases again, becoming infinite at about $\sqrt{2}\zeta_2$, after which it takes an infinite positive value and falls rapidly till it becomes equal to the velocity of light. Thus there are two centres, one at about $\frac{1}{2}\zeta_0$, and another at about $\sqrt{2}\zeta_2$ from the oscillator (more exact values are given below), from or towards which the waves appear to move with varying velocities. In most cases the $\frac{1}{2}\zeta_0$ centre would be far too near the oscillator for experiment, but the $\sqrt{2}\zeta_2$ centre, and even the point of negative velocity less than that of light, appear to be well within the field of experiment, and actually within the area inside which HERTZ made a good many experiments. Considering the complex character of the velocity of this wave of electrical disturbance in the equatorial plane, we feel doubtful as to what interpretation can possibly be put on HERTZ's interference experiments in this plane, especially on those made at a moderate distance from the oscillator.

The following figure gives the velocity curve of the equatorial electric wave *diagrammatically*. The approximate values of the chief quantities concerned have been calculated in the same manner as those on pp. 173–176, but it does not seem necessary to reproduce the calculations.



Now it is clear, both from this diagram and from that on p. 177, that we must go to $\frac{2\pi r}{\lambda} = 6$ about to get fairly constant velocity of transmission. But λ in HERTZ's experiments was about 9·6 metres (see p. 162), or the experiments, supposing a constant velocity of transmission desirable, ought to have been made at some 9 metres or more from the oscillator. HERTZ's first series are at less than 8, his third series at less than 4, and his second, which do go up to 12, he states "required rather an effort."* Within these limits the exact nature of the interference and the points at which it may be expected to occur seem open to some criticism, even in addition to the difficulties which have been raised by the problem of "multiple resonance."† The views expressed above will be strengthened by an examination of the diagram giving the phases of the waves, the velocities of which have been already discussed.

(13.) The phases have been plotted from the formulæ given above, as follows :—

I. *Transverse Component Electric Wave and Wave of Total Force perpendicular to Axis* (ϕ_1). $\delta_1 = \zeta - \beta_1$.

The formula is (xli). For the special oscillator for which our results are plotted,

$$\zeta_0 = \sin 2\chi = \cdot 126{,}8098, \qquad c_0 = \cdot 995{,}964,$$
$$\chi = \cdot 063{,}5760, \qquad c_2^2 = 2\cdot 972{,}821.$$

The asymptote is $\delta_1 = \zeta - 3\cdot 2687$ (*i.e.*, $\delta_1 = \zeta - (\pi + 2\chi)$).

We see from the diagram that the phase does not approach very rapidly to its asymptotic value, being still about 5 per cent. of its value from it, when $\zeta = 10$.

We notice that δ_1 starts from zero at the origin, and after high contact becomes small and negative. This negative portion of the phase is too small to be seen on the large diagram, but an enlarged inset figure is added. The curve cuts the axis and becomes positive at about 1·165, or at about 1·42 metres from the origin, a distance within which some of HERTZ's experiments on interference were made. The range of negative phase could be considerably increased with a more rapidly damped oscillator.

The approximate value‡ of Oa for any value of χ is

$$= (15 \tan\chi)^{\frac{1}{3}} + \frac{20}{7}\tan\chi + \frac{2000}{1176}\tan\chi\left(\frac{\tan^2\chi}{15}\right)^{\frac{1}{3}} + \dots \quad \text{(xliii)}.$$

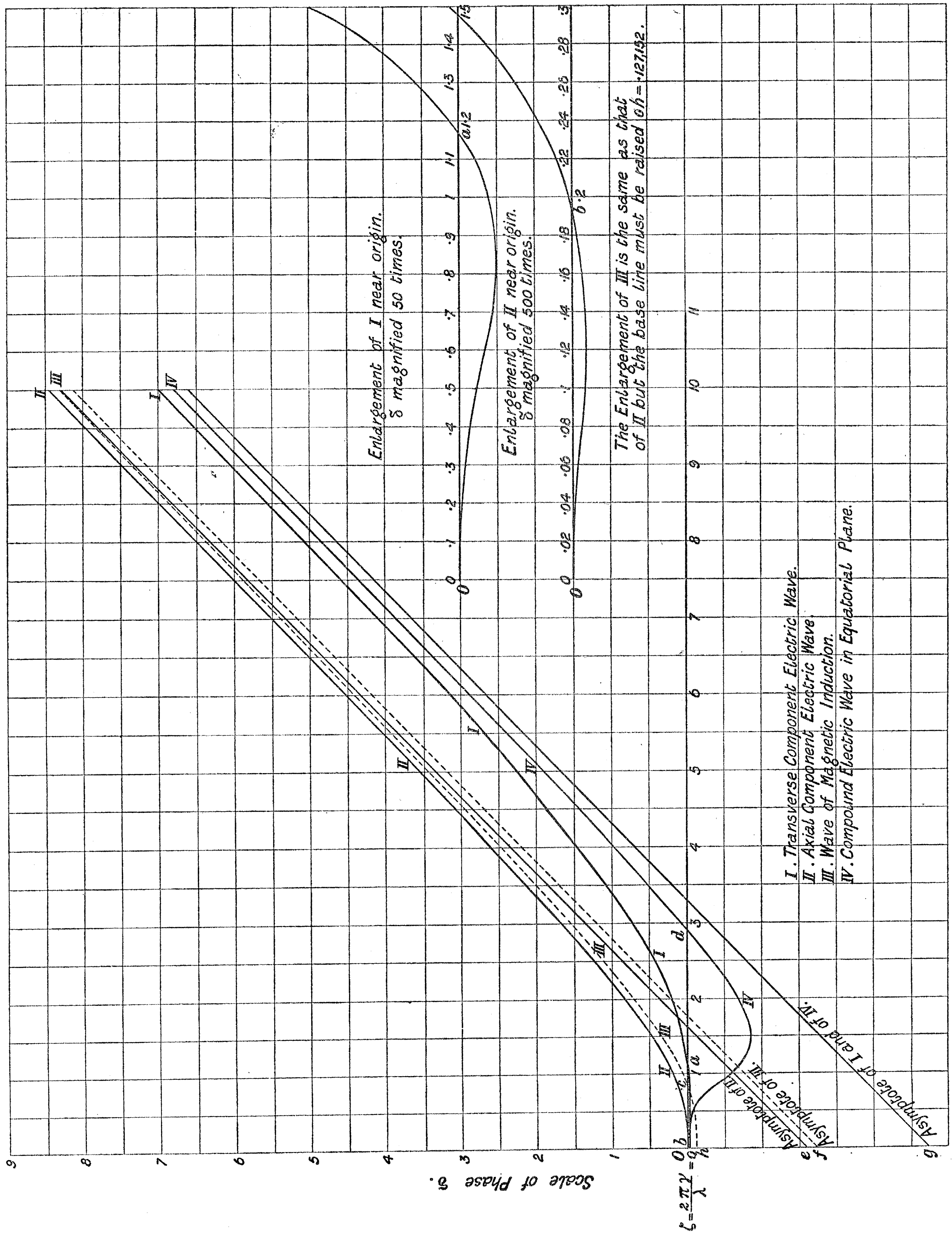
II. *Axial Component Electric Wave and Wave of Total Radial Force* (ϕ_2). $\delta_2 = \zeta - \beta_2$.

In this case the formula used was deduced from (xxiv). This may be altered to

* 'Electric Waves,' pp. 118, 120, and 119 respectively.

† POINCARÉ, 'Électricité et Optique,' vol. 2, p. 195 *et seq.*, and p. 249 *et seq.*

‡ We owe this general solution to the kindness of Mr. L. N. G. FILON, M.A.


Enlargement of I near origin.
δ magnified 50 times.
Enlargement of II near origin.
δ magnified 500 times.
The Enlargement of III is the same as that
of II but the base line must be raised oh = ·127132.
I. Transverse Component Electric Wave.
II. Axial Component Electric Wave.
III. Wave of Magnetic Induction.
IV. Compound Electric Wave in Equatorial Plane.
Asymptote of I and of IV
Asymptote of III
Asymptote of II
Scale of Phase δ.

$$\tan(\beta_2 - \chi) = \frac{\zeta - \frac{1}{2}\zeta_0}{\cos^2 \chi} \quad \text{(xliv)},$$

which serves readily for logarithmic calculation.

The portion of the curve for which δ is negative now lies between 0 and ·2, and we must multiply it some 500 times to render the small negative values of δ visible.

The approximate value of O*b* for any value of χ is given by

$$Ob = \tfrac{3}{2}\zeta_0,$$

and thus = ·19021 for our special case.

The asymptote is given by the 45° line

$$\delta = \zeta - \left(\frac{\pi}{2} + \chi\right),$$

or, for our special case O*e* = 1·63437.

The curve for this wave has a far less sensible negative portion, and approaches much more rapidly to its asymptote, being in our special case at a distance less than 2 per cent. of the value of δ from its asymptote when $\zeta = 10$.

Thus we see that there is a marked difference between the phases of the component transverse and component axial electric waves. The first appears to proceed from a centre at distance O*a* given by (xliii) from the centre, and the second from a distance O*b* equal $\frac{3}{2}\zeta_0$, or practically from a point so close to the centre of the oscillator as not to be sensible in the case of any but very rapidly damped wave trains. The ultimate phase difference of the two waves $= \frac{\pi}{2} + \chi$.

III. *Wave of Magnetic Induction.* $\delta_3 = \zeta - \beta_3$.

Here we can either use the graphical method of p. 179, or the simple relation $\delta_3 = \delta_2 - 2\chi$.

The latter method shows us that the asymptote is

$$\delta = \zeta - \left(\frac{\pi}{2} + 3\chi\right),$$

or in our special case

$$\delta = \zeta - 1{\cdot}7615.$$

Thus the ultimate difference of phase between the transverse component electric and the magnetic waves is $\frac{1}{2}\pi - \chi$, and between the axial electric and magnetic waves is -2χ.

It will be seen that this wave of magnetic induction for a damped wave differs considerably from that given by HERTZ.

The most notable difference is the *finite negative* value of δ_3 at the origin, and this negative value of the phase continues until $\frac{2\pi r}{\lambda} = Oc$; for the general case we may take approximately

$$Oc^* = (6\chi)^{\frac{1}{3}}\left(1 + \tfrac{1}{15}(6\chi)^{\frac{2}{3}}\right).$$

For the special value of χ used above, *i.e.*, $\chi = 3° \ 38' \ 33{\cdot}5''$, we find

$$Oc = {\cdot}8664.$$

This is at a distance from the source of

$$r = {\cdot}1379\lambda,$$

or about 1·324 metres. Thus we see that the phase of this action does not, as HERTZ states ('Electric Waves,' p. 154), "increase continuously from the origin itself." In discussing the velocity we have seen that $d\beta_3/dr = d\beta_0/dr$, so that the velocity of propagation is identical with that given by (xxi). Thus it becomes indefinitely great when $2\pi r/\lambda = \sin 2\chi = {\cdot}1268$ for the above values of the constants. The point, therefore, of infinitely great velocity is much closer to the origin than that of zero phase. We think HERTZ considered these two points to be identical. At any rate, he held that the wave moved with infinite velocity up to the point of zero phase. For he argues from his conclusion that the phase increases continuously from the origin that:

"The phenomena which point to a finite rate of propagation must in the case of these interferences† make themselves felt even close to the oscillator. This was indeed apparent in the experiments, and therein lay the advantage of this kind of interference. But, contrary to the experiment, the apparent velocity near to the oscillator comes out greater than at a distance from it . . ."‡

As a matter of fact, the alterations in the velocity of transmission of either magnetic wave or transverse component electric wave are very complex in the neighbourhood of the oscillator, and these variations do not bear a simple relation to the points of zero phase, from which points HERTZ assumes the outward moving wave to start. Something of the difficulties HERTZ met with in his experiments on "interferences of the second kind" made close to the oscillator may well have been due to this complexity of result arising from the use of a damped wave-train.

IV. *The Compound Electric Wave in the Equatorial Plane, and the Wave of Total Transverse Electric Force* $(\phi_1 + \phi_2)$. $\delta_4 = \zeta - \beta_4$.

The formula is now (xl).

For the special oscillator for which our curve is drawn, $c_0 = {\cdot}995{,}9635$, $c_1^2 = {\cdot}988{,}928$, $\zeta_0 = {\cdot}126{,}8098$, and the asymptote is $\delta = \zeta - 3{\cdot}2687$.

* This must only be taken as the basis to find a second approximation, as the series converges slowly.

† Interferences of the "second kind": see 'Electric Waves,' p. 119.

‡ 'Electric Waves,' p. 154.

The general value for Od, giving the point of zero phase, is Od = 2·743,707 + ·112,193 ζ_0, approximately. For our special case, Od = 2·9079.

It will be seen that our curve approaches closer to its asymptote than the curve for the true wave of transverse vibrations. Further, the centre, d, from which the wave with positive velocity may be supposed to start, is moved far further from the origin. It is rather further from the origin than it would be in the case of an undamped train.* There seems, therefore, considerable doubt as to what HERTZ was really measuring in the case of his interference experiments in the equatorial plane. Was he really measuring IV., or possibly I., obscured in its effect by the superposition of II.? It is, we think, needful to re-examine the whole matter physically, endeavouring to distinguish between the components ϕ_1 and ϕ_2. For this purpose the direction of the axis, rather than the plane of the equator, seems the more suitable for experiment. For in this direction the magnetic induction and the transverse electric component wave disappear, and we have the axial component electric wave only to deal with, *i.e.*, II. only, I. and III. being not extant.

Further, the numerous theoretical singular points to which we have referred seem to deserve, if possible, physical investigation. To do this we require to emphasise as much as possible their distance from the oscillator. But this means that we must get from a small oscillator a long wave-length, if the rate of damping (χ) be fixed, or, if the wave-length be fixed, we require very rapid damping. Neither of these conditions seem easy of physical realisation. Still something might be done by way of striking a balance, and, at any rate, we must not disregard the fact that a considerable portion of HERTZ's experiments were made inside that portion of the field where the influence of these singular points and of the damping should at least theoretically make themselves felt.

(14.) *Conclusions.*—We may draw the following general conclusions :—

(i.) The effect of damping makes itself very sensible in modifying the form of the wave-surface as propagated into space from a theoretical oscillator. The typical Hertzian wave-diagrams require to be replaced by the fuller series accompanying this memoir.

(ii.) *Three* waves of electro-magnetic force may be considered as sent out from the oscillator, and these waves we believe capable of physical identification.

First, a component wave of transverse electric force, determined by ϕ_1. This also gives the wave-speed and phase of force perpendicular to the oscillator axis.

Secondly, a component wave of electric force parallel to the axis, determined by ϕ_2. This also gives the wave-speed and phase of force radial to the oscillator.

Thirdly, a wave of magnetic force.

* In this case Od = 2·743,707 correct to the last place. In the general approximate value given above, we have neglected terms of the order χ^2, and the approximation is not nearly as accurate as this.

The waves of magnetic force and of component axial electric force move outwards each with the same velocity at all points, and this velocity is equal for all points at the same distance from the oscillator. The intensity of the first force for points on the same sphere varies as the cosine of the latitude, but that of the second force is constant. The wave of component transverse electric force moves outwards with equal velocity for all points at the same distance from the oscillator, and its amplitude varies as the cosine of the latitude. Its velocity after it has reached a certain distance from the origin, is always greater than that of the waves of component axial electric force, and of magnetic force, and its excess over the velocity of light tends to become three times the excess of the velocity of the wave of magnetic force over the velocity of light.

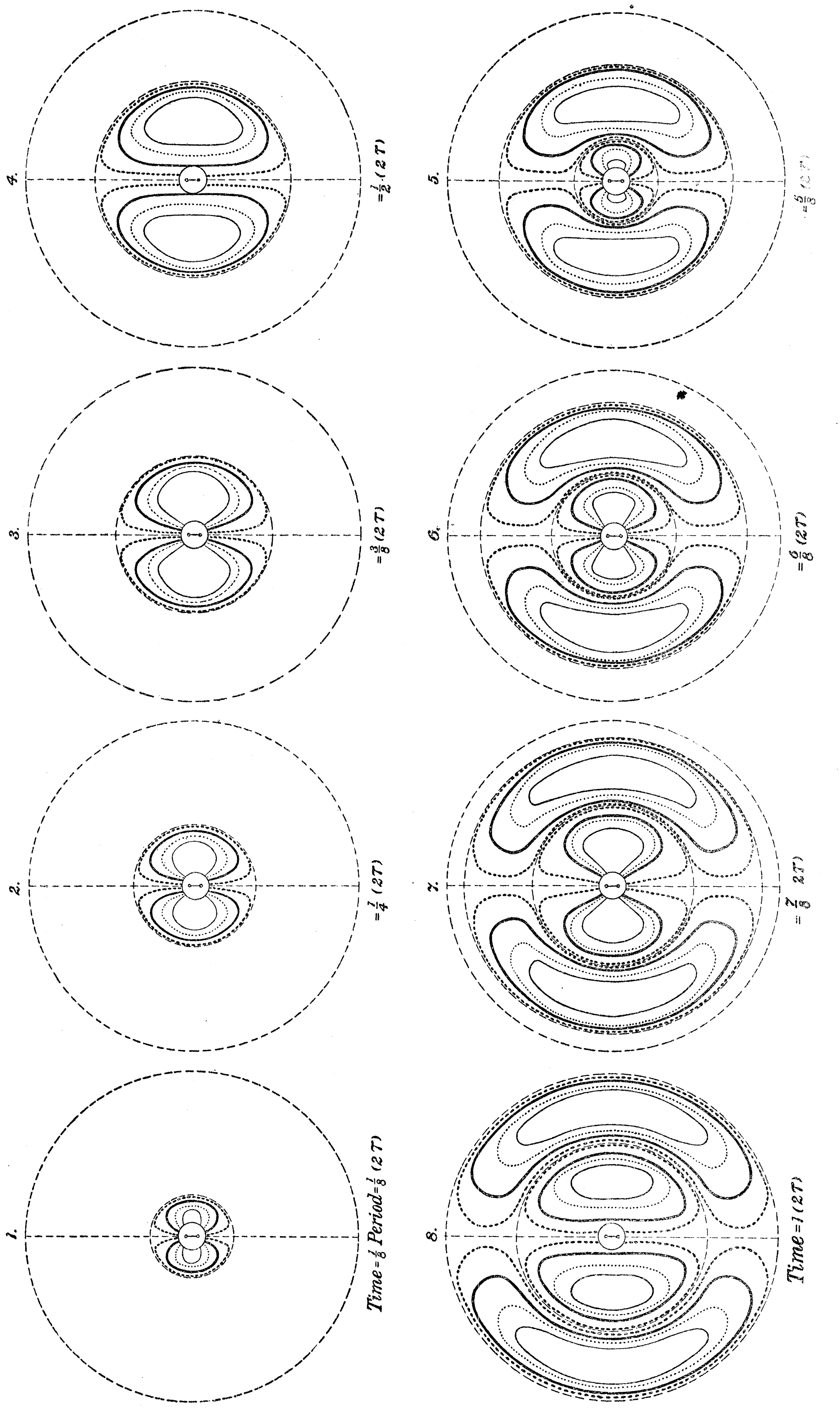
(iii.) The velocities of these waves undergo remarkable changes in the neighbourhood of the oscillator, but still at distances such as HERTZ experimented at, and which seem indeed to some extent within the field of possible physical investigation.

(iv.) The point of zero phase for both transverse and axial component electric waves does not coincide with the centre of the oscillator, so that these waves appear to start from a sphere of small but finite radius round the oscillator. A fourth wave dealt with by HERTZ, the wave of magnetic induction, does not, as he supposes, start from the centre of the oscillator with zero phase, but in the case of a damped wave train with a small but finite phase.

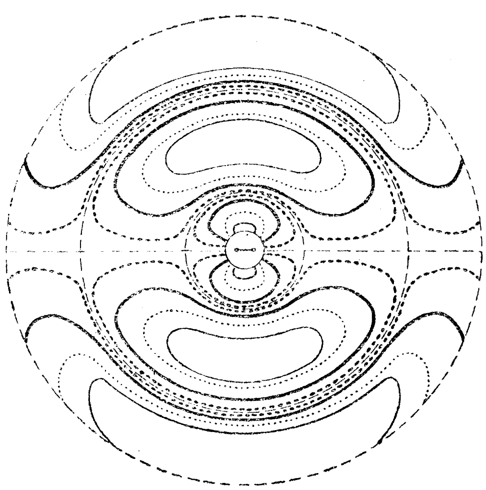
(v.) Our analysis of these waves and of their singular points in the neighbourhood of the oscillator appears to add something to HERTZ's discussion; it is possible that it may throw light on the difficulties which arise in connecting with some of his interference experiments. It would seem to us that all interference experiments ought to be made at distances greater than 6 to 7 $(\lambda/2\pi)$ from the centre of the oscillator, roughly about a wave-length from the oscillator, whereas HERTZ rather terminated than started his experiments at this distance. At such distances the phase curves are approximately parallel to their asymptotes.

Finally, we are not unaware of the physical difficulties attending experiments, at such distances, and wish in conclusion to again emphasise the fact that our analysis only applies to a *theoretical* type of oscillator. It is, however, the type for which HERTZ himself endeavoured to provide a mathematical investigation.

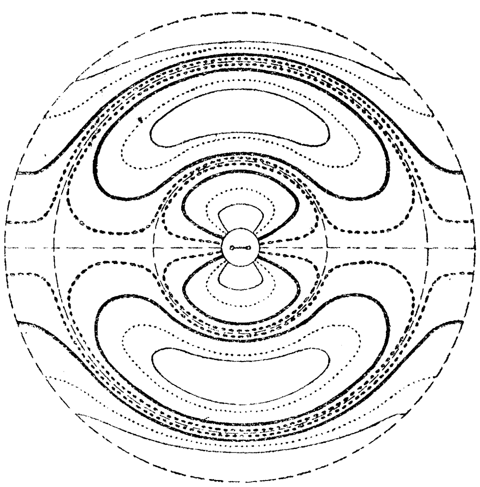
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES



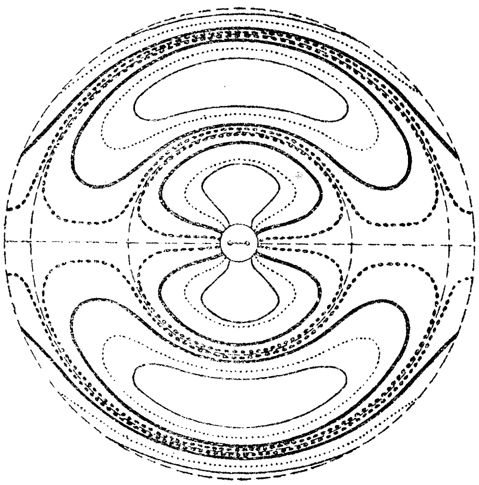
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES



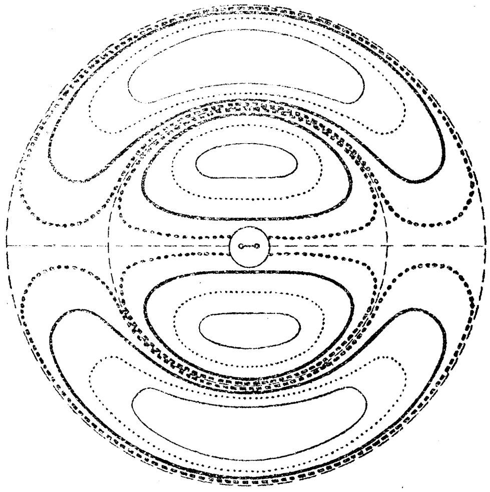
9. *Time* $= 1\frac{1}{8}\,(2T)$



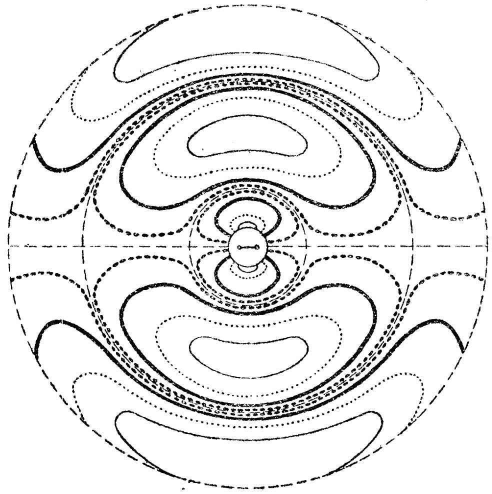
10. $= 1\frac{1}{4}\,(2T)$



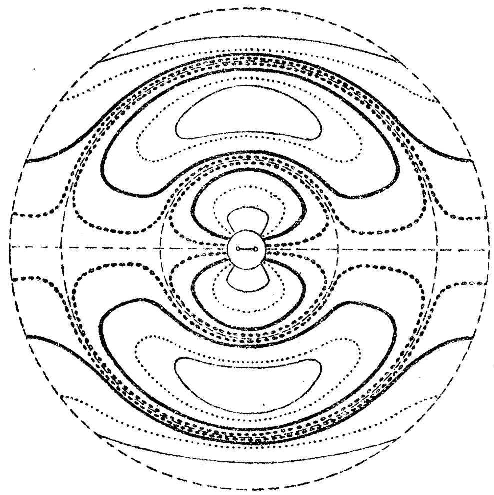
11. $= 1\frac{3}{8}\,(2T)$



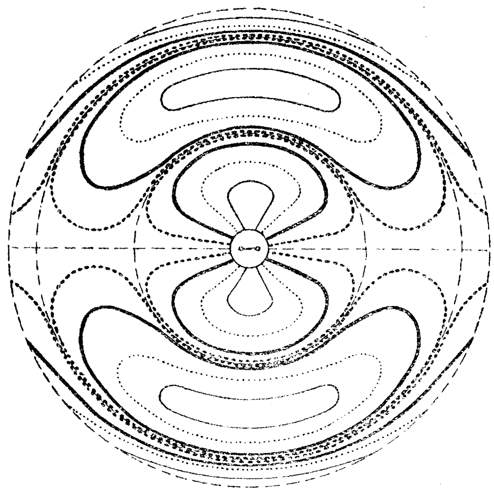
12. $= 1\frac{1}{2}\,(2T)$



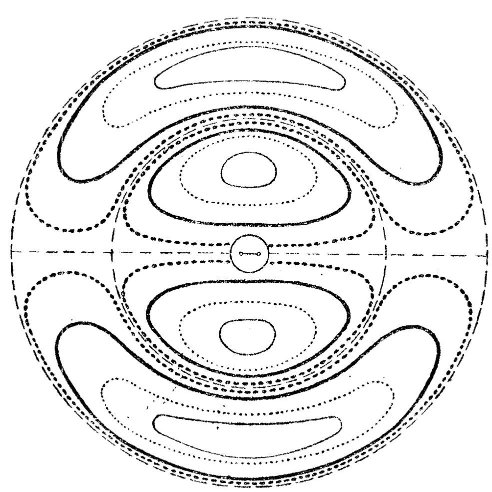
13. $= 1\frac{5}{8}\,(2T)$



14. $= 1\frac{3}{4}\,(2T)$



15. $= 1\frac{7}{8}\,(2T)$



16. *Time* $= 2\,(2T)$

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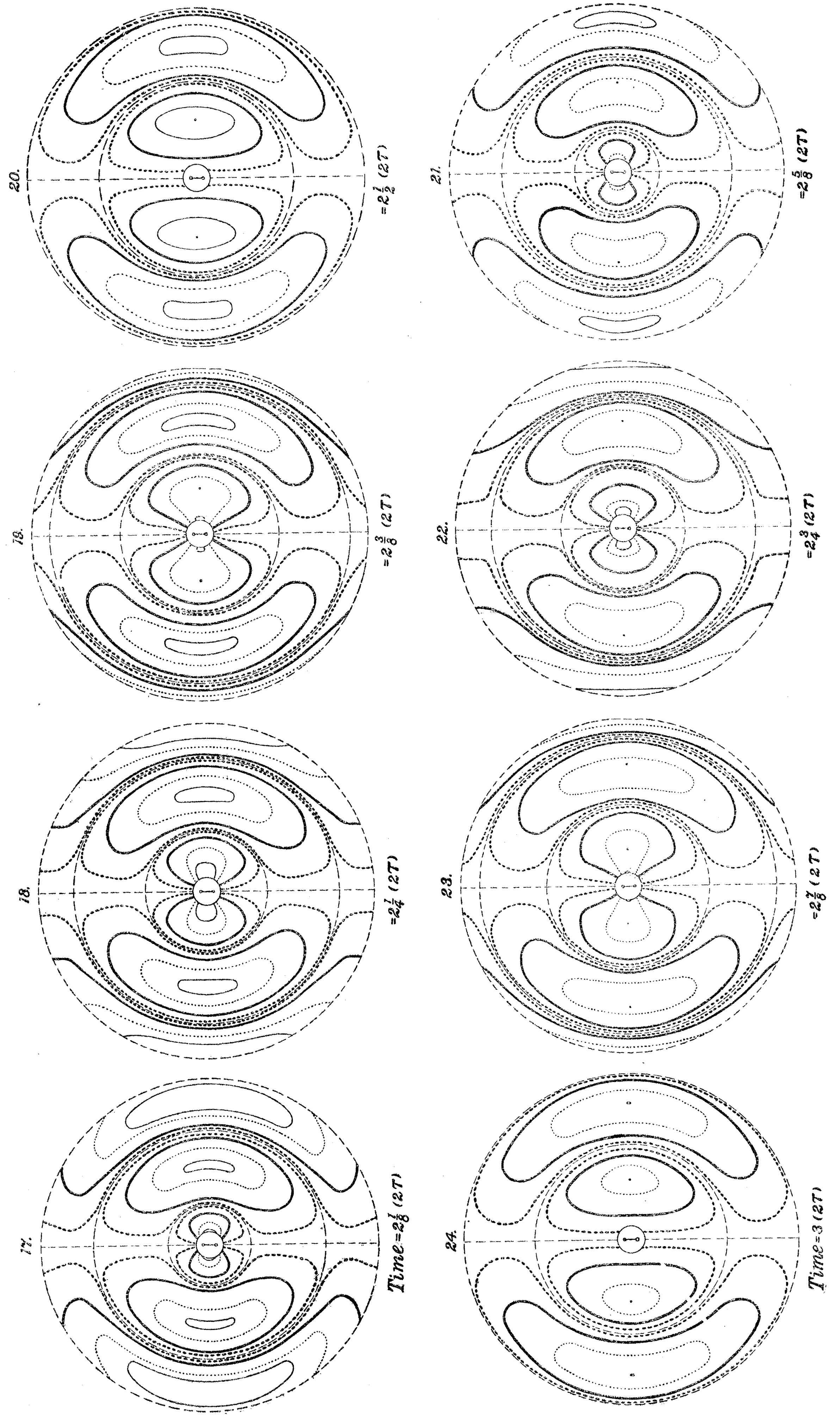
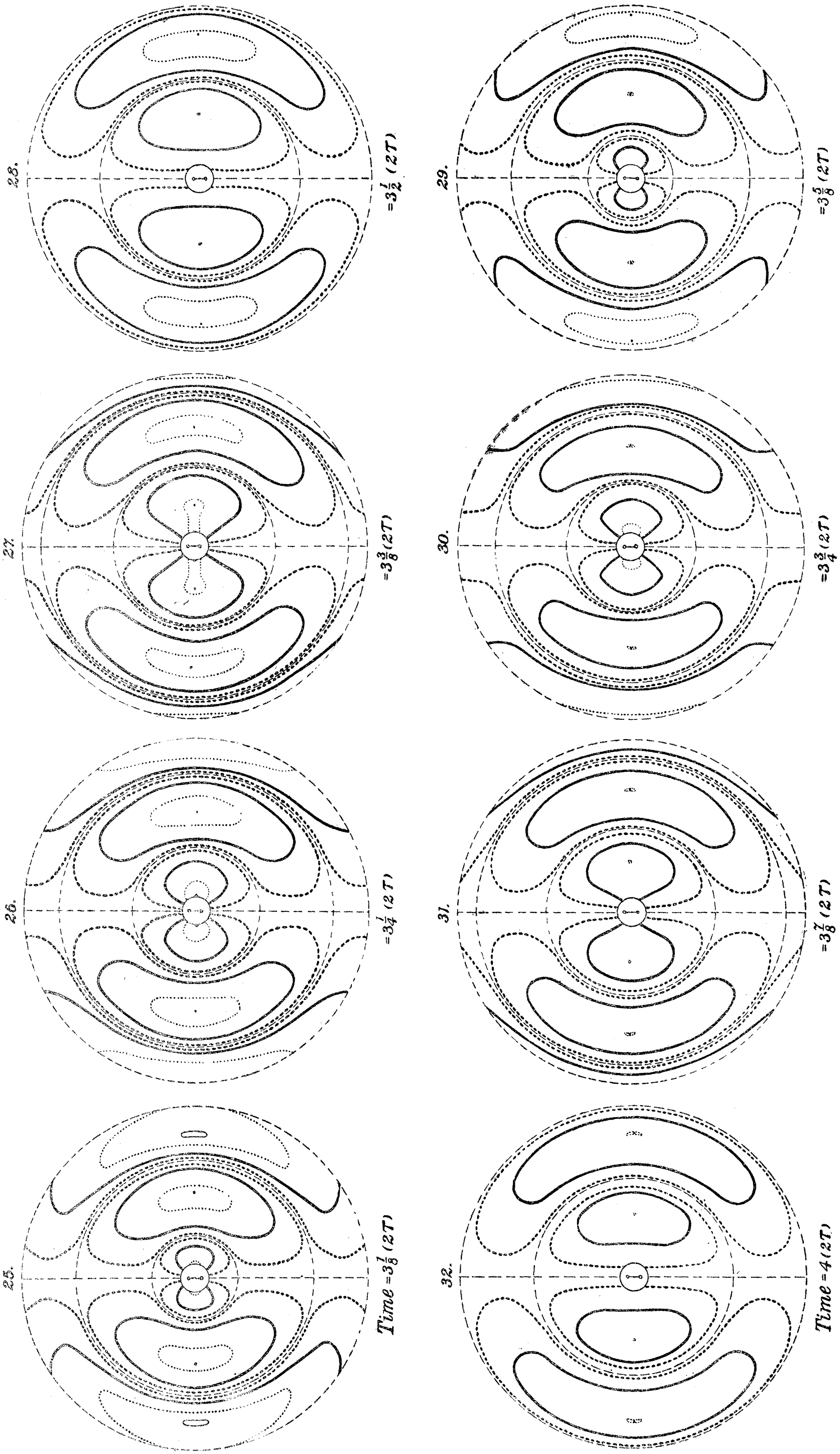
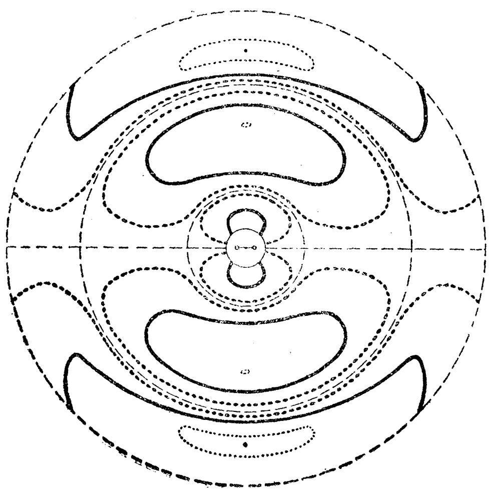

20.
=$2\frac{1}{2}$ (2T)
21.
=$2\frac{5}{8}$ (2T)
19.
=$2\frac{3}{8}$ (2T)
22.
=$2\frac{3}{4}$ (2T)
18.
=$2\frac{1}{4}$ (2T)
23.
=$2\frac{7}{8}$ (2T)
17.
Time=$2\frac{1}{8}$ (2T)
24.
Time=3 (2T)



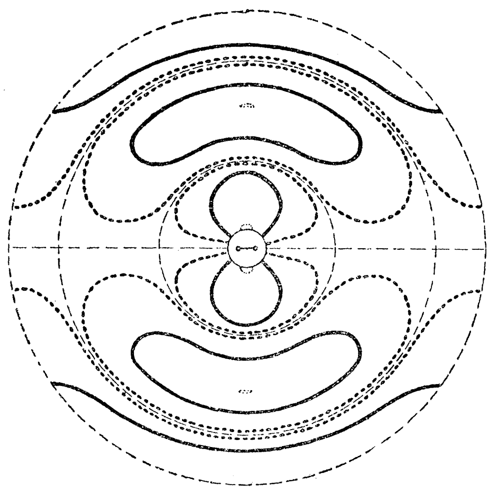
28.
$= 3\frac{1}{2}$ (2T)
29.
$= 3\frac{5}{8}$ (2T)
27.
$= 3\frac{3}{8}$ (2T)
30.
$= 3\frac{3}{4}$ (2T)
26.
$= 3\frac{1}{4}$ (2T)
31.
$= 3\frac{7}{8}$ (2T)
25.
Time $= 3\frac{1}{8}$ (2T)
32.
Time = 4 (2T)

33.



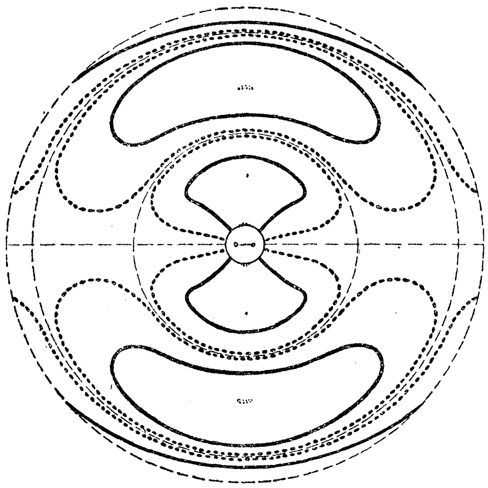
Time $=4\frac{1}{8}(2T)$

34.



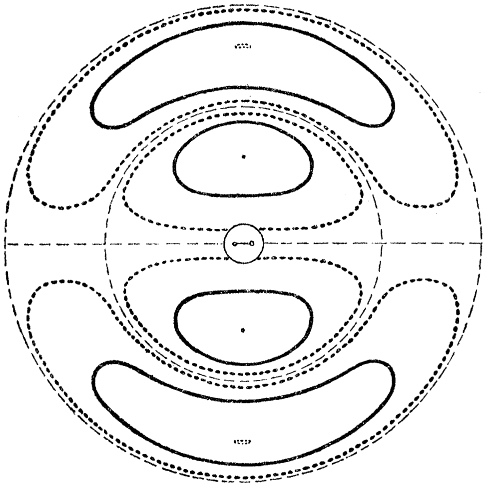
$=4\frac{1}{4}(2T)$

35.



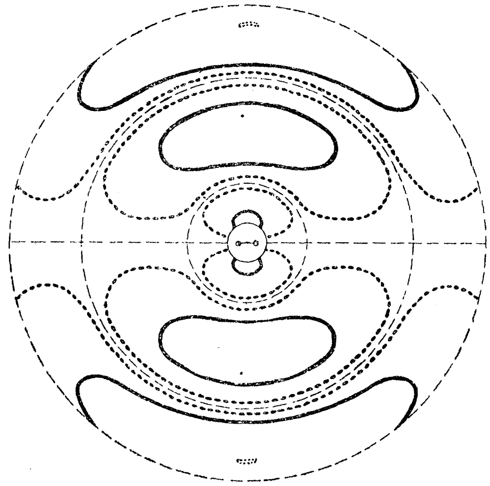
$=4\frac{3}{8}(2T)$

36.



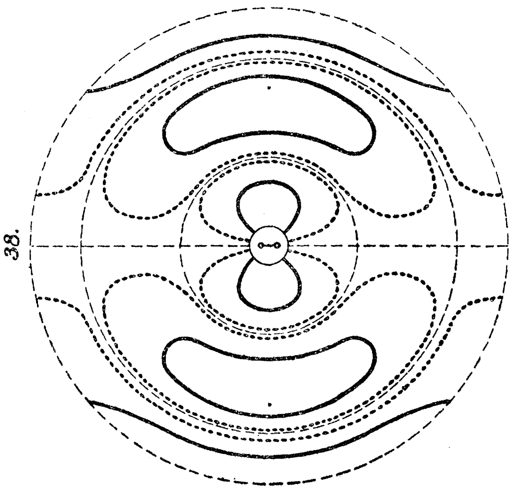
$=4\frac{1}{2}(2T)$

37.



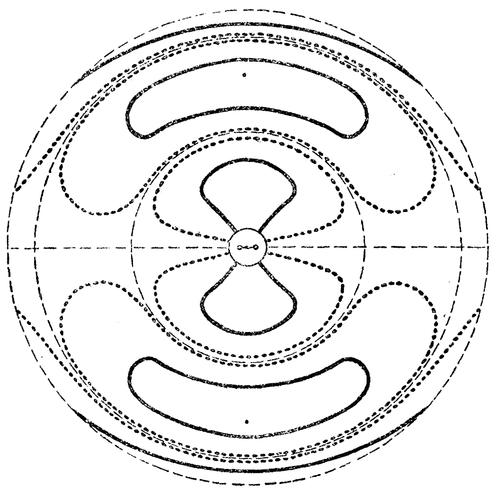
$=4\frac{5}{8}(2T)$

38.



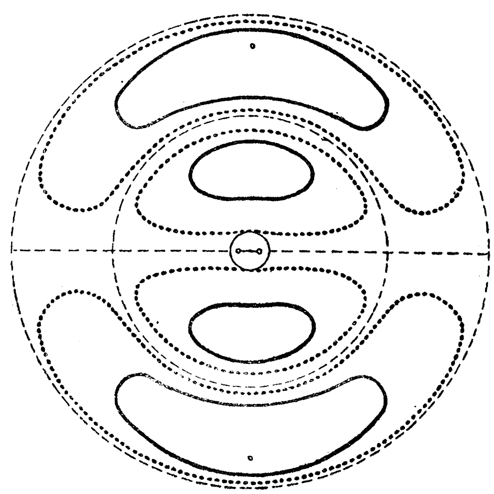
$=4\frac{3}{4}(2T)$

39.



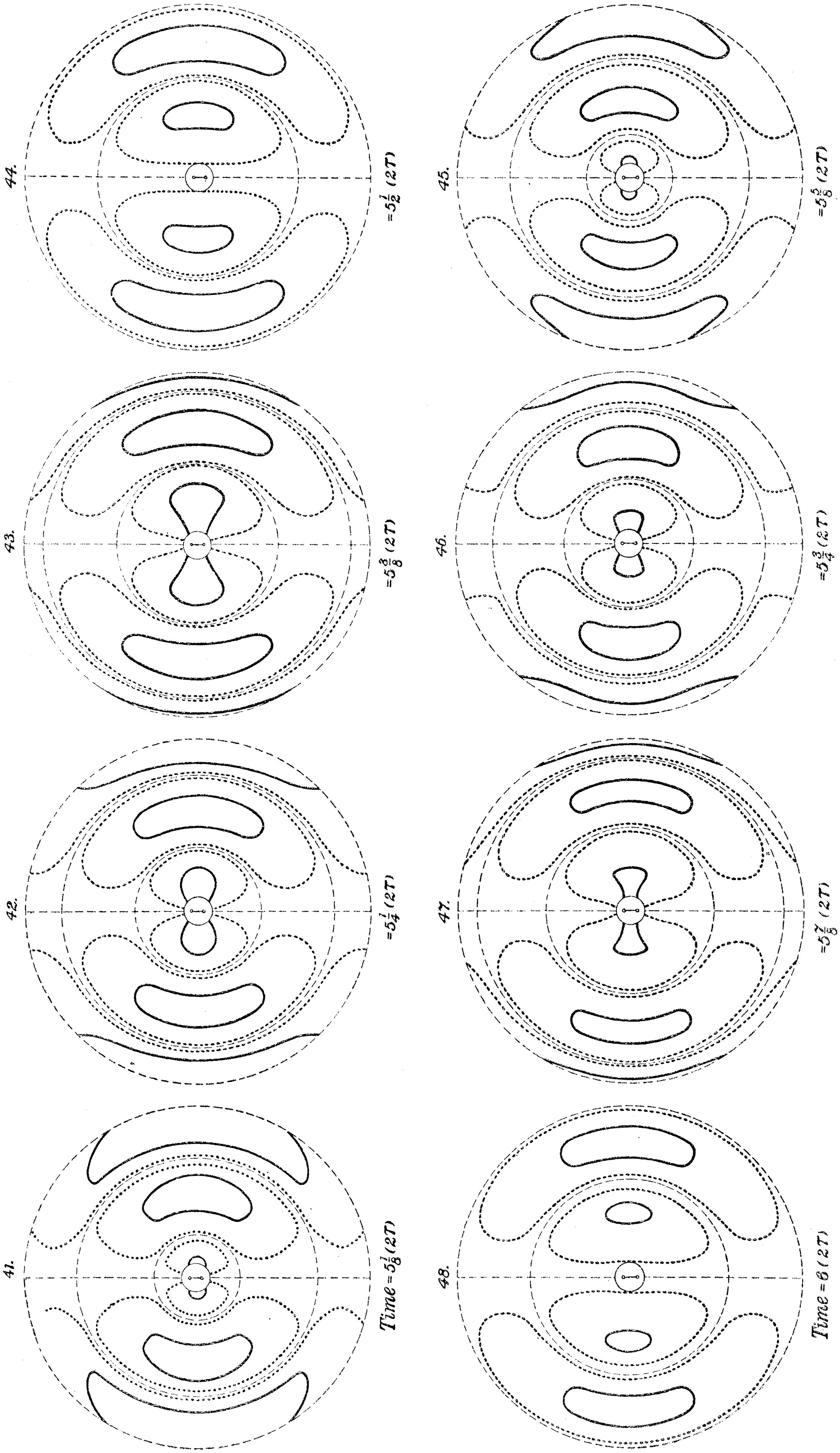
$=4\frac{7}{8}(2T)$

40.

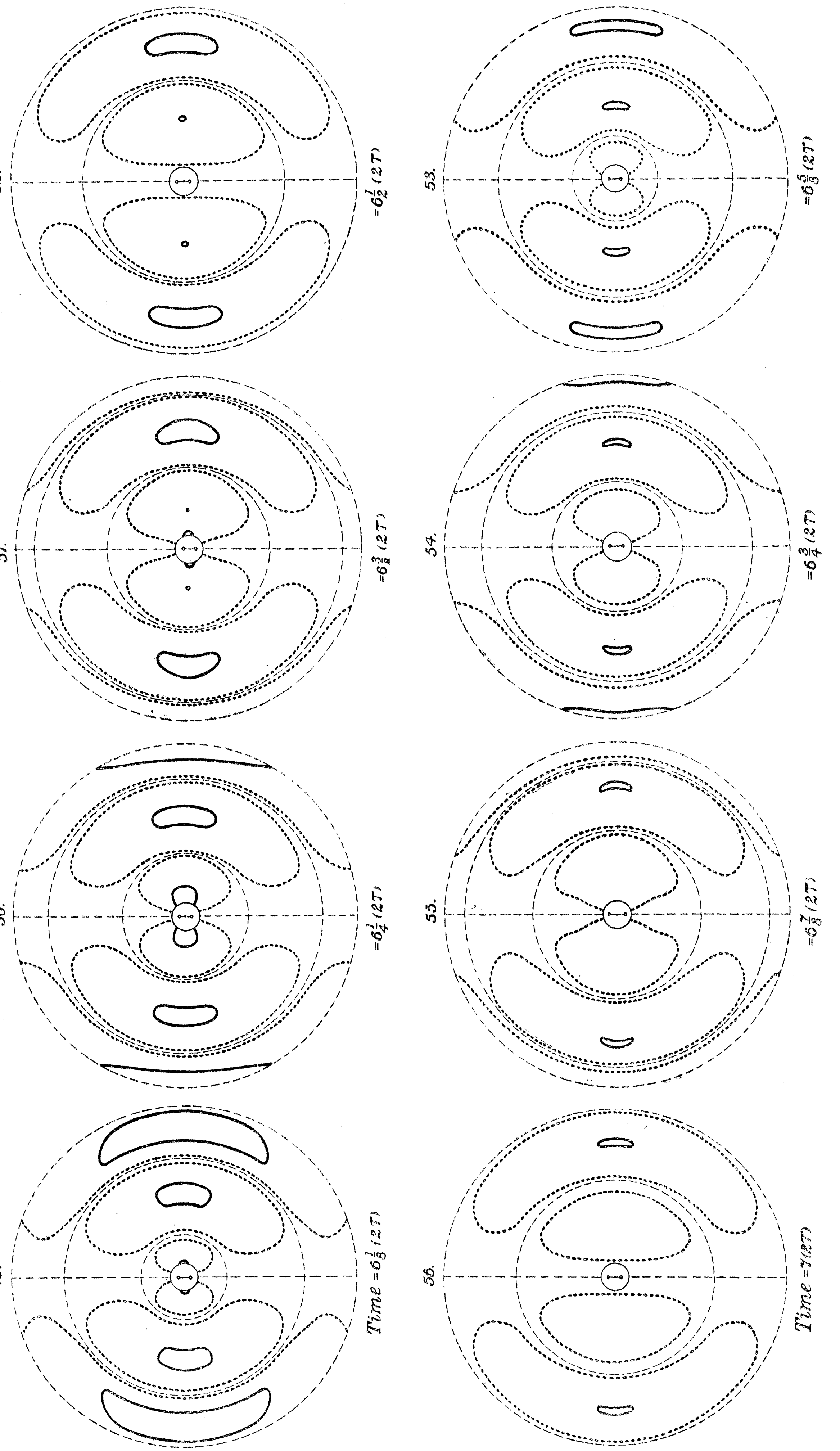


Time $=5(2T)$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES



PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES



49.
Time = $6\frac{1}{8}$ (2T)
50.
= $6\frac{1}{4}$ (2T)
51.
= $6\frac{3}{8}$ (2T)
52.
= $6\frac{1}{2}$ (2T)
53.
= $6\frac{5}{8}$ (2T)
54.
= $6\frac{3}{4}$ (2T)
55.
= $6\frac{7}{8}$ (2T)
56.
Time = 7 (2T)